Section 7: Electromagnetic Radiation

7.1 Radiation

We discussed the propagation of plane electromagnetic waves through various media, but we were not interested in a way of how the waves were generated. Like all electromagnetic fields, their source is some arrangement of electric charge. But a charge at rest does not generate electromagnetic waves; nor does a steady current. The waves are due to *accelerating* charges, and *changing* currents. Here we consider how such charges and currents produce electromagnetic waves – that is, how they ***radiate***.

The signature of radiation is irreversible flow of energy away from the source. We assume that the source is localized near the origin. If we now imagine a gigantic spherical shell, out at radius r, the total power passing out through this surface is the integral of Poynting's vector:

$$P = \lim_{r\to\infty} P(r) = \lim_{r\to\infty} \oint \mathbf{S}\cdot\mathbf{n}da = \frac{1}{\mu_0}\lim_{r\to\infty}\oint\left(\mathbf{E}\times\mathbf{B}\right)\cdot\mathbf{n}da \tag{7.1}$$

The power *radiated* is the limit of this quantity as r goes to infinity. This is the energy (per unit time) that is transported out to infinity, and never comes back. Now, the area of the sphere is $4\pi r^2$, so for radiation to occur Poynting's vector must decrease (at large r) no faster than $1/r^2$ (if it went like $1/r^3$, for example, then $P(r)$ would go like $1/r$, and P would be zero). According to Coulomb's law, electrostatic fields fall off like $1/r^2$ (or even faster, if the total charge is zero), and the Biot-Savart law says that magnetostatic fields go like $1/r^2$ (or faster), which means that $S \sim 1/r^4$, for static configurations. So *static* sources do not radiate. But Jefimenko's equations indicate that *time-dependent* fields include terms that go like $1/r$; it is *these* terms that are responsible for electromagnetic radiation.

The study of radiation, then, involves picking out the parts of **E** and **B** that go like $1/r$ at large distances from the source, constructing from them the $1/r^2$ term in S, integrating over a large spherical surface, and taking the limit as $r\to\infty$.

We consider a general situation, in which electromagnetic radiation is produced by an *arbitrary distribution of charges and currents,* with an *arbitrary time dependence* (not necessarily oscillating with a single frequency ω). Our only restrictions are that

(i) the source is confined to a bounded region V of space,

(ii) the charges are moving slowly.

These conditions will allow us to formulate useful approximations for the behavior of the electric and magnetic fields.

To make the slow-motion approximation precise and to define different electromagnetic field zones, we introduce the following scaling quantities:

r_s – characteristic length scale of the charge and current distribution,

t_s – characteristic time scale over which the distribution changes,

$v_s = \dfrac{r_s}{t_s}$ – characteristic velocity of the charge motion in the source,

$\omega_s = \dfrac{2\pi}{t_s}$ – characteristic frequency of the charge motion in the source,

$\lambda_s = \dfrac{2\pi c}{\omega_s} = ct_s$ – characteristic wavelength of radiation.

The characteristic length scale is defined such that the distribution of charge and current is localized within a region whose volume is of the order of r_s^3. The characteristic time scale is defined such that $\partial\rho/\partial t$ is of order ρ/t_s throughout the source.

The slow-motion approximation means that $v_s = r_s / t_s$ is much smaller than the speed of light:

$$v_s \ll c . \tag{7.2}$$

This condition gives us $r_s = v_s t_s \ll c t_s = \lambda_s$, or

$$r_s \ll \lambda_s . \tag{7.3}$$

The source is therefore confined to a region that is *much smaller* than a typical wavelength of the radiation.

There are three spatial regions of interest:

The near (static) zone: $r \ll \lambda_s$. (7.4)

The intermediate (induction) zone: $r_s \ll r \sim \lambda_s$. (7.5)

The far (radiation) zone: $r_s \ll \lambda_s \ll r$. (7.6)

We will see that the fields have very different properties in these zones. In the near zone the fields have the character of static fields, with radial components and variation with the distance that depend in detail on the properties of the source. In the far zone, on the other hand, the fields are transverse to the radius vector and fall of as 1/*r* which is typical for radiation fields. We will compute the potentials and fields in the near and far zones.

7.2 Electric dipole radiation

We begin by calculating the scalar potential,

$$\Phi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|} d^3r' , \tag{7.7}$$

where V is the region of space occupied by the source and $t_r = t - |\mathbf{r}-\mathbf{r}'|/c$ is the retarded time. In the *near zone*, we can treat $|\mathbf{r}-\mathbf{r}'|/c$ as a small quantity and Taylor-expand the charge density about the current time *t*. We have

$$\rho(t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}) = \rho(t) - \frac{1}{c}\dot{\rho}(t)|\mathbf{r}-\mathbf{r}'| + \ldots \tag{7.8}$$

where an overdot indicates differentiation with respect to *t*. Relative to the first term, the second term is of order $r/(ct_s) = r/\lambda_s$, and by virtue of Eq. (7.4), this is small in the near zone. The third term would be smaller still, and we neglect it. We then have

$$\Phi(\mathbf{r},t) \approx \frac{1}{4\pi\varepsilon_0}\left[\int_V \frac{\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|} d^3r' - \frac{1}{c}\int_V \dot{\rho}(\mathbf{r}',t) d^3r'\right] = \frac{1}{4\pi\varepsilon_0}\left[\int_V \frac{\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|} d^3r' - \frac{1}{c}\frac{d}{dt}\int_V \rho(\mathbf{r}',t) d^3r'\right], \tag{7.9}$$

The second term vanishes, because it involves the time derivative of the total charge $\int_V \rho(\mathbf{r}',t) d^3r'$ which is conserved. We have therefore obtained

$$\Phi(\mathbf{r},t) \approx \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|} d^3r' . \tag{7.10}$$

We see that the near-zone potential is similar to its usual static expression, except the fact that charge density depends on time. The time delay between the source and the potential has disappeared, and what we have is a potential that adjusts instantaneously to the changes within the distribution. The electric field it produces is then a "time-changing electrostatic field". This near-zone field does not behave as radiation.

To witness radiative effects, we must go to the *radiation zone.* Here $|\mathbf{r}-\mathbf{r}'|$ is large and we can no longer Taylor-expand the density as we did previously. Instead, we must introduce another approximation technique. We use the fact that in the induction zone, r is much larger than r', so that

$$|\mathbf{r}-\mathbf{r}'| = r - \hat{\mathbf{r}}\cdot\mathbf{r}' + \dots . \tag{7.11}$$

This gives

$$\rho\left(t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right) \approx \rho\left(t - \frac{r}{c} + \frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\right) = \rho\left(t_0 + \frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\right), \tag{7.12}$$

where we defined *retarded time at the origin*

$$t_0 \equiv t - \frac{r}{c} . \tag{7.13}$$

Let us now Taylor-expand the charge density about the retarded time t_0 instead of the current time t. We have

$$\rho\left(t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right) \approx \rho\left(t_0 + \frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\right) = \rho(t_0) + \dot{\rho}(t_0)\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c} + \dots \tag{7.14}$$

where an overdot now indicates differentiation with respect to t_0.

Inside the integral for Φ, we approximate

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r} + \dots . \tag{7.15}$$

It is sufficient to keep only the leading term $1/r$ because the higher order terms do not contribute to radiation. The radiation zone potential is therefore

$$\begin{aligned}\Phi(\mathbf{r},t) &\approx \frac{1}{4\pi\varepsilon_0 r}\left\{\int_V \rho(\mathbf{r}',t_0)d^3r' + \int_V \dot{\rho}(\mathbf{r}',t_0)\left(\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\right)d^3r'\right\} = \\ &= \frac{1}{4\pi\varepsilon_0 r}\left\{\int_V \rho(\mathbf{r}',t_0)d^3r' + \frac{\hat{\mathbf{r}}}{c}\cdot\frac{d}{dt_0}\int_V \rho(\mathbf{r}',t_0)\mathbf{r}'d^3r'\right\}.\end{aligned} \tag{7.16}$$

In the first integral we recognize the total charge of the distribution:

$$q = \int_V \rho(\mathbf{r}',t_0)d^3r' . \tag{7.17}$$

It is independent of t_0 by virtue of charge conservation. In the second integral we recognize the dipole moment vector of the charge distribution:

$$\mathbf{p}(t_0) = \int_V \rho(\mathbf{r}',t_0)\mathbf{r}'d^3r' . \tag{7.18}$$

This does depend on retarded time t_0 because the charge density is time dependent. Our final expression for the potential is therefore

$$\Phi(\mathbf{r},t) \approx \frac{1}{4\pi\varepsilon_0}\frac{q}{r} + \frac{1}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}\cdot\dot{\mathbf{p}}(t_0)}{cr}. \tag{7.19}$$

The first term on the right-hand side of Eq. (7.19) in the static, monopole potential associated with the total charge q. This term does not depend on time and is *not* associated with the propagation of radiation; we shall simply omit it in later calculations. The second term, on the other hand, is radiative: it depends on retarded time t_0 and decays as $1/r$. We see that the radiative part of the scalar potential is produced by a time-changing dipole moment of the charge distribution; it is nonzero whenever $d\mathbf{p}/dt$ is nonzero.

An exact expression for the vector potential is

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|}d^3r'. \tag{7.20}$$

In the *near zone*, we approximate the current density as follows

$$\mathbf{J}(\mathbf{r}',t_r) \approx \mathbf{J}(\mathbf{r}',t), \tag{7.21}$$

and we obtain

$$\mathbf{A}(\mathbf{r},t) \approx \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}d^3r'. \tag{7.22}$$

We see that here the vector potential takes its static form. The potential responds virtually instantaneously to changes in the distribution, and there are no radiative effects in the near zone.

In the *radiation zone*, we have instead

$$\mathbf{J}(t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}) \approx \mathbf{J}(t_0 + \frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}) = \mathbf{J}(t_0) + \dot{\mathbf{J}}(t_0)\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c} + \ldots \tag{7.23}$$

For now, we will keep only the first term in this equation, then

$$\mathbf{A}(\mathbf{r},t) \approx \frac{\mu_0}{4\pi r}\int \mathbf{J}(\mathbf{r}',t_0)d^3r'. \tag{7.24}$$

In static situations, the volume integral of $\mathbf{J}$ vanishes. But here the current density depends on time, and we have instead

$$\int \mathbf{J}(\mathbf{r}',t_0)d^3r' = \dot{\mathbf{p}}(t_0). \tag{7.25}$$

To prove this, we write the i component of $\dot{\mathbf{p}}(t)$ as follows

$$\begin{aligned} \dot{p}_i(t) &= \frac{d}{dt}\int_V \rho(\mathbf{r},t)x_i d^3r = \int_V \frac{\partial\rho}{\partial t}x_i d^3r = -\int_V (\nabla\cdot\mathbf{J})x_i d^3r = \\ &= -\int_V \nabla\cdot(\mathbf{J}x_i)d^3r + \int_V \mathbf{J}\cdot(\nabla x_i)d^3r = -\oint_S \mathbf{J}\cdot\mathbf{n}x_i da + \int_V \mathbf{J}\cdot\hat{\mathbf{x}}_i d^3r = \int_V J_i d^3r. \end{aligned} \tag{7.26}$$

Here, we took into account the continuity equation, $\partial\rho/\partial t = -\nabla\cdot\mathbf{J}$, and the fact that no current is crossing surface S bounding volume V. The vector potential is therefore

$$\mathbf{A}(\mathbf{r},t) \approx \frac{\mu_0}{4\pi}\frac{\dot{\mathbf{p}}(t_0)}{r}. \tag{7.27}$$

This has the structure of a spherical wave, and we see that the radiative part of the vector potential is produced by a time-changing dipole moment.

The potentials

$$\Phi(\mathbf{r},t) \approx \frac{1}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}\cdot\dot{\mathbf{p}}(t_0)}{cr}, \tag{7.28}$$

$$\mathbf{A}(\mathbf{r},t) \approx \frac{\mu_0}{4\pi}\frac{\dot{\mathbf{p}}(t_0)}{r}, \tag{7.29}$$

are generated by time variations of the dipole moment vector $\mathbf{p}$ of the charge and current distribution. They therefore give rise to *electric-dipole radiation,* the leading-order contribution, in our slow-motion approximation, to the radiation emitted by an arbitrary source. We now compute the electric and magnetic fields in this approximation.

To obtain the electric field we keep only those terms that decay as $1/r$, and neglect terms that decay faster. For example, when computing the gradient of the scalar potential we can neglect $\nabla r^{-1} = -\hat{\mathbf{r}}/r^2$ so that

$$\begin{aligned}\nabla\Phi(\mathbf{r},t) &\approx \frac{1}{4\pi\varepsilon_0}\frac{1}{cr}\nabla\left[\hat{\mathbf{r}}\cdot\dot{\mathbf{p}}(t_0)\right] = \frac{1}{4\pi\varepsilon_0}\frac{1}{cr}\sum_i \frac{\partial\left[\hat{\mathbf{r}}\cdot\dot{\mathbf{p}}(t_0)\right]}{dx_i}\hat{\mathbf{x}}_i = \frac{1}{4\pi\varepsilon_0}\frac{1}{cr}\sum_i \frac{\partial\left[\hat{\mathbf{r}}\cdot\dot{\mathbf{p}}(t_0)\right]}{\partial t_0}\frac{\partial t_0}{\partial x_i}\hat{\mathbf{x}}_i = \\ &= \frac{1}{4\pi\varepsilon_0}\frac{1}{cr}\left[\hat{\mathbf{r}}\cdot\ddot{\mathbf{p}}(t_0)\right]\nabla t_0 = \frac{1}{4\pi\varepsilon_0}\frac{1}{cr}\left[\hat{\mathbf{r}}\cdot\ddot{\mathbf{p}}(t_0)\right]\left(-\frac{1}{c}\right)\nabla r = -\frac{1}{4\pi\varepsilon_0}\frac{1}{c^2 r}\left(\hat{\mathbf{r}}\cdot\ddot{\mathbf{p}}\right)\hat{\mathbf{r}}.\end{aligned} \tag{7.30}$$

The electric field is then

$$\mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t} = \frac{1}{4\pi\varepsilon_0}\frac{1}{c^2 r}\left(\hat{\mathbf{r}}\cdot\ddot{\mathbf{p}}\right)\hat{\mathbf{r}} - \frac{\mu_0}{4\pi}\frac{\ddot{\mathbf{p}}}{r} = \frac{\mu_0}{4\pi r}\left[\left(\hat{\mathbf{r}}\cdot\ddot{\mathbf{p}}\right)\hat{\mathbf{r}} - \ddot{\mathbf{p}}\right] = \frac{\mu_0}{4\pi r}\left[\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\ddot{\mathbf{p}}\right)\right]. \tag{7.31}$$

Notice that the radiation-zone electric field behaves as a spherical wave, and that it is transverse to $\hat{\mathbf{r}}$, the direction in which the wave propagates.

To get the magnetic field we need to compute $\nabla\times\mathbf{A}(\mathbf{r},t)$. Similar to (7.30), we can write

$$\mathbf{B}(\mathbf{r},t) = \nabla\times\mathbf{A}(\mathbf{r},t) \approx \frac{\mu_0}{4\pi r}\nabla\times\dot{\mathbf{p}}(t_0) = -\frac{\mu_0}{4\pi rc}\left[\hat{\mathbf{r}}\times\ddot{\mathbf{p}}(t_0)\right]. \tag{7.32}$$

This latter equality can be seen from

$$\left[\nabla\times\dot{\mathbf{p}}(t_0)\right]_i = \sum_{jk}\varepsilon_{ijk}\frac{\partial\dot{p}_k(t_0)}{\partial x_j} = -\frac{1}{c}\sum_{jk}\varepsilon_{ijk}\ddot{p}_k\frac{x_j}{r} = -\frac{1}{c}\left[\hat{\mathbf{r}}\times\ddot{\mathbf{p}}\right]_i. \tag{7.33}$$

Notice that the radiation-zone magnetic field behaves as a spherical wave, and that it is orthogonal to both $\hat{\mathbf{r}}$ and the electric field. Notice finally that the fields are in phase – they both depend on $\ddot{\mathbf{p}}$ and are related as follows

$$\mathbf{E} = c\left(\mathbf{B}\times\hat{\mathbf{r}}\right). \tag{7.34}$$

so that their magnitudes are $|\mathbf{B}|/|\mathbf{E}| = 1/c$.

7.3 Energy radiated

Poynting's vector is

$$\mathbf{S} = \frac{1}{\mu_0}\left(\mathbf{E}\times\mathbf{B}\right) = \frac{1}{\mu_0}\left[c\left(\mathbf{B}\times\hat{\mathbf{r}}\right)\times\mathbf{B}\right] = \frac{c}{\mu_0}\left[B^2\hat{\mathbf{r}} - \left(\mathbf{B}\cdot\hat{\mathbf{r}}\right)\mathbf{B}\right] = \frac{c}{\mu_0}B^2\hat{\mathbf{r}}. \tag{7.35}$$

The fact that Poynting's vector is directed along $\hat{\mathbf{r}}$ indicates that the electromagnetic field energy travels along with the wave.

The energy crossing a sphere of radius r per unit time is given by $P = \oint \mathbf{S}\cdot\mathbf{n}da$, where $\mathbf{n}da = r^2 d\Omega\hat{\mathbf{r}}$ and $d\Omega = \sin\theta d\theta d\phi$. Substituting Eqs. (7.35) and (7.32) yields

$$P = \oint \mathbf{S}\cdot\mathbf{n}da = \frac{\mu_0}{(4\pi)^2 c}\oint\left[\hat{\mathbf{r}}\times\ddot{\mathbf{p}}\right]^2 d\Omega\,. \tag{7.36}$$

To evaluate the integral, we use the trick of momentarily aligning the z axis with the instantaneous direction of $\ddot{\mathbf{p}}$ – we must do this for each particular value of t_0. Then $\hat{\mathbf{r}}\times\ddot{\mathbf{p}} = |\ddot{\mathbf{p}}|\sin\theta$ and

$$P = \frac{\mu_0}{(4\pi)^2 c}|\ddot{\mathbf{p}}|^2\oint\sin^2\theta d\Omega = \frac{\mu_0}{(4\pi)^2 c}|\ddot{\mathbf{p}}|^2\, 2\pi\frac{4}{3} = \frac{\mu_0}{6\pi c}|\ddot{\mathbf{p}}|^2\,. \tag{7.37}$$

This is the total power radiated by a slowly-moving distribution of charge and current. The angular distribution of power is described by

$$\frac{dP}{d\Omega} = \frac{\mu_0}{(4\pi)^2 c}|\ddot{\mathbf{p}}|^2\sin^2\theta\,. \tag{7.38}$$

Example: Center-fed linear antenna

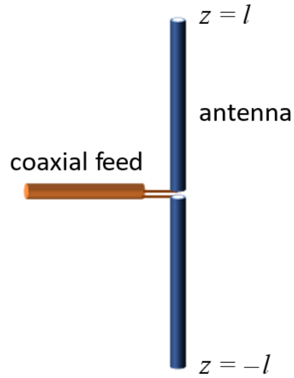

Fig. 7.1 Center-fed linear antenna. The oscillating current is provided by a coaxial feed.

As an example of a radiating system, we consider a thin wire of total length $2l$ which is fed an oscillating current through a small gap at its midpoint. The wire runs along the z axis, from $z = -l$ to $z = l$, and the gap is located at $z = 0$ (Fig. 7.1). For such antennas, the current typically oscillates both in time and in space, and it is usually represented by

$$\mathbf{J}(\mathbf{r},t) = I_m\sin\left[k(l-|z|)\right]\delta(x)\delta(y)\hat{\mathbf{z}}\cos\omega t\,, \tag{7.39}$$

where $k = \omega/c$. The current is an even function of z (it is the same in both arms of the antenna) and it goes to zero at both ends (at $z = \pm l$). The amplitude of the current in the gap (at $z = 0$) is $I_0 = I_m\sin(kl)$.

We want to calculate the total power radiated by this antenna, using the electric-dipole approximation. To be consistent, we must ensure that $v_s \ll c$ for this current distribution, that is $l \equiv r_s \ll \lambda_s \equiv 2\pi/k$. In other words, we must demand that $kl \ll 1$, which means that $k|z|$ is small throughout the antenna. We can therefore approximate $\sin\left[k(l-|z|)\right]$ by $k(l-|z|)$ and Eq. (7.39) becomes

$$\mathbf{J}(\mathbf{r},t) = I_0(1-|z|/l)\delta(x)\delta(y)\hat{\mathbf{z}}\cos\omega t\ , \tag{7.40}$$

where $I_0 = I_m kl$ is the value of the current at the gap. In this approximation, the current no longer oscillates in space: it simply goes from its peak value I_0 at the gap to zero at the two ends of the wire.

To compute the power radiated by our simplified antenna, we first need to calculate $p(t)$, the second time derivative of the dipole moment vector. For this it is efficient to turn to Eq. (7.25),

$$\dot{\mathbf{p}}(t) = \int \mathbf{J}(\mathbf{r},t)d^3r\ , \tag{7.41}$$

in which we substitute Eq. (7.40). We then have

$$\dot{\mathbf{p}}(t) = I_0\hat{\mathbf{z}}\cos\omega t\int_{-l}^{l}(1-|z|/l)dz\ . \tag{7.42}$$

Evaluating the integral gives

$$\dot{\mathbf{p}}(t) = -I_0 l\hat{\mathbf{z}}\cos\omega t\ . \tag{7.43}$$

Taking the second derivative yields

$$\ddot{\mathbf{p}}(t) = I_0 l\omega\hat{\mathbf{z}}\sin\omega t = \left(I_0 c\right)\left(kl\right)\hat{\mathbf{z}}\sin\omega t\ . \tag{7.44}$$

By introducing a vector θ between $\hat{\mathbf{r}}$ and $\hat{\mathbf{z}}$, we find for the angular distribution of the power

$$\frac{dP}{d\Omega} = \frac{\mu_0}{\left(4\pi\right)^2 c}\left(I_0 c\right)^2\left(kl\right)^2\sin^2\omega t\sin^2\theta\ . \tag{7.45}$$

After averaging over a complete wave cycle, this reduces to

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{\mu_0 c}{32\pi^2}\left(I_0 kl\right)^2\sin^2\theta\ . \tag{7.46}$$

To obtain the total power radiated we must integrate over the angles. Using $\oint\sin^2\theta d\Omega = 8\pi/3$, we arrive at our final result:

$$\left\langle P\right\rangle = \frac{\mu_0 c}{12\pi}\left(I_0 kl\right)^2\ . \tag{7.47}$$

For a fixed frequency ω, the power increases like the square of the feed current I_0. For a fixed current, the power increases like the square of the frequency, so long as the condition $kl \ll 1$ is satisfied. From Eq. (7.46) we learn that most of the energy is radiated in the directions perpendicular to the antenna; none of the energy propagates along the axis.

7.4 Magnetic dipole and electric quadrupole radiation

Electric-dipole radiation corresponds to the leading order approximation of the electromagnetic field in the in an expansion in powers of v_s/c, where v_s is a typical internal velocity of charges in the source. In some cases, however, the electric dipole moment **p** either vanishes or does not depend on time, and the leading term is actually zero. In such cases, we need to compute the next term in the expansion. We will see that in this case the electromagnetic radiation is determined by the magnetic dipole moment **m** or/and the electric quadrupole moment tensor Q_{ij}. These quantities are defined as follows:

$$\mathbf{p}(t) = \int_V \rho(\mathbf{r}',t)\mathbf{r}'d^3r'\ , \tag{7.48}$$

$$\mathbf{m}(t) = \frac{1}{2}\int_V \mathbf{r}' \times \mathbf{J}(\mathbf{r}',t) d^3r' , \tag{7.49}$$

$$Q_{ij}(t) = \int_V \rho(\mathbf{r}',t)\left(3x_i'x_j' - r'^2\delta_{ij}\right) d^3r' , \tag{7.50}$$

where V is the volume that contains the charge and current distributions; this volume is bounded by the surface S.

To make calculations more efficient, we first derive a few useful results. We can anticipate that in the radiation zone the potentials will have the form of a spherical wave. For example, the vector potential is given by

$$\mathbf{A}(\mathbf{r},t) \approx \frac{\mu_0}{4\pi}\frac{\mathbf{w}(\hat{\mathbf{r}},t_0)}{r}, \tag{7.51}$$

where $t_0 = t - r/c$ and $\mathbf{w}$ is a vector that will be determined. We will not need the expression for the scalar potential: as we will see, in the radiation zone $\mathbf{E}$ can be directly obtained from $\mathbf{B}$ that will allows us to calculate the radiated power.

Given the form of the vector potential, its curl is

$$\mathbf{B}(\mathbf{r},t) = \nabla\times\mathbf{A}(\mathbf{r},t) \approx \frac{\mu_0}{4\pi r}\nabla\times\mathbf{w}(\hat{\mathbf{r}},t_0) \approx \frac{\mu_0}{4\pi r}\nabla t_0 \times \frac{\partial \mathbf{w}(\hat{\mathbf{r}},t_0)}{\partial t_0} = -\frac{\mu_0}{4\pi}\frac{\hat{\mathbf{r}}\times\dot{\mathbf{w}}}{cr} . \tag{7.52}$$

Here we neglected the terms of the order of $1/r^2$ (these terms come from the gradient of $1/r$ and from the explicit dependence of $\mathbf{w}$ on $\hat{\mathbf{r}}$).

In order to find the electric field, we go back to the Maxwell's equation:

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J} + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} . \tag{7.53}$$

Since the current density vanishes outside the volume V, we have $\mathbf{J} = 0$ in the radiation zone, and therefore

$$\frac{\partial\mathbf{E}}{\partial t} = c^2\nabla\times\mathbf{B} == -\frac{\mu_0 c}{4\pi}\nabla\times\left(\frac{\hat{\mathbf{r}}\times\dot{\mathbf{w}}}{r}\right) \approx \frac{\mu_0 c}{4\pi}\left(\nabla\times\dot{\mathbf{w}}\right)\times\frac{\hat{\mathbf{r}}}{r} \approx -\frac{\mu_0}{4\pi}\frac{\left(\hat{\mathbf{r}}\times\ddot{\mathbf{w}}\right)\times\hat{\mathbf{r}}}{r} . \tag{7.54}$$

Integrating with respect to t gives

$$\mathbf{E}(\mathbf{r},t) \approx -\frac{\mu_0}{4\pi}\frac{\left(\hat{\mathbf{r}}\times\dot{\mathbf{w}}\right)\times\hat{\mathbf{r}}}{r} = c\mathbf{B}(\mathbf{r},t)\times\hat{\mathbf{r}} . \tag{7.55}$$

Note that the constant of integration can be neglected because it is time-independent and therefore does not contribute to electromagnetic radiation. Poynting's vector is therefore

$$\mathbf{S} = \frac{1}{\mu_0}\left(\mathbf{E}\times\mathbf{B}\right) = \frac{c}{\mu_0}B^2\hat{\mathbf{r}} = \frac{\mu_0}{16\pi^2 c}\frac{\left|\hat{\mathbf{r}}\times\dot{\mathbf{w}}\right|^2}{r^2}\hat{\mathbf{r}} . \tag{7.56}$$

Evaluating this on a sphere of constant radius r results in

$$dP = \mathbf{S}\cdot\mathbf{n}da = \frac{\mu_0}{16\pi^2 c}\left|\hat{\mathbf{r}}\times\dot{\mathbf{w}}\right|^2 d\Omega , \tag{7.57}$$

or

$$\frac{dP}{d\Omega} = \frac{\mu_0}{16\pi^2 c} \left|\hat{\mathbf{r}} \times \dot{\mathbf{w}}(\hat{\mathbf{r}}, t_0)\right|^2, \tag{7.58}$$

where $d\Omega = \sin\theta d\theta d\phi$ is an element of the solid angle. Eq. (7.58) gives the angular distribution of the radiation power, which is controlled by the dependence of **w** on the radial vector $\hat{\mathbf{r}}$. Notice that the fields and the radiation's angular profile depend only on $\hat{\mathbf{r}} \times \dot{\mathbf{w}}$, and that they are insensitive to the component of **w** along $\hat{\mathbf{r}}$.

From there results we infer that irrespective to the exact expression for **w**, the electric and magnetic fields are transverse and mutually orthogonal, that their amplitudes differ by a factor of *c*, that they are in phase, and that the field energy travels in the radiation direction.

Vector potential in the radiation zone

We are now ready to calculate vector **w**. We start from an exact expression for the vector potential

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} d^3 r'. \tag{7.59}$$

In this we substitute approximations:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} \approx \frac{1}{r}. \tag{7.60}$$

$$t_r = t - \frac{|\mathbf{r} - \mathbf{r}'|}{c} \approx t - \frac{r}{c} + \frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{c} = t_0 + \frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{c}. \tag{7.61}$$

where $t_0 = t - r/c$ is the retarded time at the origin. By expanding the current density in Taylor series, we find in the radiation zone

$$\mathbf{J}(t - \frac{|\mathbf{r} - \mathbf{r}'|}{c}) \approx \mathbf{J}(t_0 + \frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{c}) = \mathbf{J}(t_0) + \dot{\mathbf{J}}(t_0) \frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{c} + \ldots \tag{7.62}$$

Relative to the leading term, the second term is smaller by a factor of order of $r_s / (ct_s) = v_s / c$. This term was neglected in our discussion of electric dipole radiation, but we will keep it now. Substituting the expansion for **J** into Eq. (7.59) gives

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi r} \left[\int \mathbf{J}(\mathbf{r}', t_0) d^3 r' + \frac{1}{c} \frac{d}{dt_0} \int \mathbf{J}(\mathbf{r}', t_0) \left(\hat{\mathbf{r}} \cdot \mathbf{r}'\right) d^3 r' \right]. \tag{7.63}$$

We have shown earlier (Eq. (7.25)) that

$$\dot{\mathbf{p}}(t) = \int \mathbf{J}(\mathbf{r}, t) d^3 r. \tag{7.64}$$

It is shown in Appendix that

$$\int \mathbf{J}(\mathbf{r}', t_0) \left(\hat{\mathbf{r}} \cdot \mathbf{r}'\right) d^3 r' = \mathbf{m}(t_0) \times \hat{\mathbf{r}} + \frac{1}{6} \dot{\mathbf{Q}}(\hat{\mathbf{r}}, t_0) + \frac{1}{6} \hat{\mathbf{r}} \frac{d}{dt} \int \rho(\mathbf{r}', t_0) r'^2 d^3 r'. \tag{7.65}$$

Here the vector $\mathbf{Q}(\hat{\mathbf{r}}, t_0)$ defined by

$$Q_i(\hat{\mathbf{r}}, t_0) = \sum_j Q_{ij}(t_0) \hat{x}_j, \tag{7.66}$$

where $\hat{x}_j \equiv x_j / r$ are the components of the unit vector $\hat{\mathbf{r}} = \mathbf{r}/r$. The quantity in the square brackets of

Eq. (7.63) defines the vector **w**. Using Eqs. (7.64) and (7.65), we therefore have

$$\mathbf{w}(\hat{\mathbf{r}},t_0)=\dot{\mathbf{p}}(t_0)+\frac{1}{c}\left[\dot{\mathbf{m}}(t_0)\times\hat{\mathbf{r}}+\frac{1}{6}\ddot{\mathbf{Q}}(\hat{\mathbf{r}},t_0)\right]+\frac{1}{6c}\hat{\mathbf{r}}\frac{d^2}{dt_0^2}\int\rho(\mathbf{r}',t_0)r'^2d^3r'. \tag{7.67}$$

The last term here is proportional to $\hat{\mathbf{r}}$. But we have seen that the electric and magnetic fields, as well as the radiated power, depend only on $\hat{\mathbf{r}}\times\dot{\mathbf{w}}$. This means that we can ignore the last term in our expression for **w** so that

$$\mathbf{w}(\hat{\mathbf{r}},t_0)=\dot{\mathbf{p}}(t_0)+\frac{1}{c}\left[\dot{\mathbf{m}}(t_0)\times\hat{\mathbf{r}}+\frac{1}{6}\ddot{\mathbf{Q}}(\hat{\mathbf{r}},t_0)\right]. \tag{7.68}$$

The first term on the right-hand side of Eq. (7.68) gives rise to electric-dipole radiation; this is the leading term in the expansion of the fields in powers of v_s/c. The second and third terms give rise to magnetic-dipole radiation and electric-quadrupole radiation, respectively; these contribute at order v_s/c beyond the leading term. To calculate the radiated power, we need

$$\hat{\mathbf{r}}\times\dot{\mathbf{w}}=\hat{\mathbf{r}}\times\ddot{\mathbf{p}}(t_0)+\frac{1}{c}\hat{\mathbf{r}}\times\left[\ddot{\mathbf{m}}(t_0)\times\hat{\mathbf{r}}+\frac{1}{6}\dddot{\mathbf{Q}}(\hat{\mathbf{r}},t_0)\right]=\hat{\mathbf{r}}\times\ddot{\mathbf{p}}(t_0)+\frac{1}{c}\left\{\ddot{\mathbf{m}}(t_0)-\left[\hat{\mathbf{r}}\cdot\ddot{\mathbf{m}}(t_0)\right]\hat{\mathbf{r}}\right\}+\frac{1}{6c}\hat{\mathbf{r}}\times\dddot{\mathbf{Q}}(\hat{\mathbf{r}},t_0). \tag{7.69}$$

The term which is responsible for magnetic dipole radiation is

$$\left[\hat{\mathbf{r}}\times\dot{\mathbf{w}}\right]_{mag.dip.}=\frac{1}{c}\left[\ddot{\mathbf{m}}-\left(\hat{\mathbf{r}}\cdot\ddot{\mathbf{m}}\right)\hat{\mathbf{r}}\right], \tag{7.70}$$

so that

$$\left|\hat{\mathbf{r}}\times\dot{\mathbf{w}}\right|^2=\frac{1}{c^2}\left[\left|\ddot{\mathbf{m}}\right|^2-\left(\hat{\mathbf{r}}\cdot\ddot{\mathbf{m}}\right)^2\right], \tag{7.71}$$

According to Eq. (7.58) the radiated power per solid angle is

$$\left(\frac{dP}{d\Omega}\right)_{mag.dip.}=\frac{\mu_0}{16\pi^2c^3}\left[\left|\ddot{\mathbf{m}}\right|^2-\left(\hat{\mathbf{r}}\cdot\ddot{\mathbf{m}}\right)^2\right]. \tag{7.72}$$

Integrating over the solid angle gives the total radiated magnetic-dipole power:

$$P_{mag.dip.}=\frac{\mu_0}{6\pi c^3}\left|\ddot{\mathbf{m}}\right|^2. \tag{7.73}$$

One might compare this result with the electric-dipole power:

$$P_{elec.dip.}=\frac{\mu_0}{6\pi c}\left|\ddot{\mathbf{p}}\right|^2. \tag{7.74}$$

In orders of magnitude, we have the electric dipole moment $p\sim\left(\rho r_s\right)r_s^3\sim\rho r_s^4$ and $\ddot{p}\sim\rho r_s^4/t_s^2$. On the other hand, $m\sim\left(jr_s\right)r_s^3\sim\rho v_s r_s^4$ and $\ddot{m}\sim\rho v_s r_s^4/t_s^2$. The ratio of powers is then

$$\frac{P_{mag.\,dip.}}{P_{elec.\,dip.}}=\left|\frac{\ddot{\mathbf{m}}}{c\ddot{\mathbf{p}}}\right|^2\sim\left(\frac{v_s}{c}\right)^2. \tag{7.75}$$

Thus, for slowing-moving sources, the power emitted in magnetic-dipole radiation is smaller than the power emitted in electric-dipole radiation by a factor of order $(v_s/c)^2$.

For electric quadrupole radiation:

$$\left|\hat{\mathbf{r}}\times\dot{\mathbf{w}}\right|^2=\frac{1}{36c^2}\left[\hat{\mathbf{r}}\times\dddot{\mathbf{Q}}(\hat{\mathbf{r}},t_0)\right]^2, \tag{7.76}$$

and the radiated power per solid angle is

$$\left(\frac{dP}{d\Omega}\right)_{el.quard.}=\frac{\mu_0}{576\pi^2c^3}\left[\hat{\mathbf{r}}\times\dddot{\mathbf{Q}}(\hat{\mathbf{r}},t_0)\right]^2. \tag{7.77}$$

It can be represented in a different way. Taking into account that

$$(\hat{\mathbf{r}}\times\mathbf{q})^2=(\hat{\mathbf{r}}\times\mathbf{q})\cdot(\hat{\mathbf{r}}\times\mathbf{q})=\hat{\mathbf{r}}\cdot\left[\mathbf{q}\times(\hat{\mathbf{r}}\times\mathbf{q})\right]=(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}})(\mathbf{q}\cdot\mathbf{q})-(\hat{\mathbf{r}}\cdot\mathbf{q})(\hat{\mathbf{r}}\cdot\mathbf{q})=|\mathbf{q}|^2-(\hat{\mathbf{r}}\cdot\mathbf{q})^2, \tag{7.78}$$

where $\mathbf{q}\equiv\dddot{\mathbf{Q}}(\hat{\mathbf{r}},t_0)$, we find

$$\left(\frac{dP}{d\Omega}\right)_{el.quard.}=\frac{\mu_0}{576\pi c^3}\left[\left|\dddot{\mathbf{Q}}\right|^2-\left(\hat{\mathbf{r}}\cdot\dddot{\mathbf{Q}}\right)^2\right]. \tag{7.79}$$

The latter equation can be written as follows

$$\left(\frac{dP}{d\Omega}\right)_{el.quard.}=\frac{\mu_0}{144\pi c^3}\sum_{ij}\dddot{Q}_i\dddot{Q}_j\left[\delta_{ij}-\hat{x}_i\hat{x}_j\right]=\sum_{ijkl}\dddot{Q}_{ik}\dddot{Q}_{jl}\left[\delta_{ij}\hat{x}_k\hat{x}_l-\hat{x}_i\hat{x}_j\hat{x}_k\hat{x}_l\right]. \tag{7.80}$$

where $\hat{x}_j\equiv x_j/r$ are components of the unit vector $\hat{\mathbf{r}}$ and we used relationship $Q_i(\hat{\mathbf{r}},t)=\sum_j Q_{ij}(t)\hat{x}_j$. Integrating over solid angle and taking into account that

$$\int\hat{x}_i\hat{x}_j d\Omega=\frac{4\pi}{3}\delta_{ij} \tag{7.81}$$

and

$$\int\hat{x}_i\hat{x}_j\hat{x}_k\hat{x}_l d\Omega=\frac{4\pi}{15}\left(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right), \tag{7.82}$$

we find

$$\begin{aligned}P_{el.quard.}&=\frac{\mu_0}{144\pi c^3}\sum_{ijkl}\dddot{Q}_{ik}\dddot{Q}_{jl}\left[\frac{1}{3}\delta_{ij}\delta_{kl}-\frac{1}{15}\left(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right)\right]=\\&=\frac{\mu_0}{144\pi c^3}\left[\frac{1}{3}\sum_{ik}\dddot{Q}_{ik}\dddot{Q}_{ik}-\frac{1}{15}\left(\sum_{ik}\dddot{Q}_{ik}\dddot{Q}_{ik}+\sum_{ij}\dddot{Q}_{ii}\dddot{Q}_{jj}+\sum_{ij}\dddot{Q}_{ij}\dddot{Q}_{ji}\right)\right]=\frac{\mu_0}{144\pi c^3}\left[\frac{1}{5}\sum_{ik}\dddot{Q}_{ik}\dddot{Q}_{ik}\right].\end{aligned} \tag{7.83}$$

Here we used the property that Q_{ij} is symmetric traceless tensor so that $\dddot{Q}_{ij}=\dddot{Q}_{ji}$ and $\sum_i\dddot{Q}_{ii}=0$. Our final result for the electric-quadrupole radiated power is

$$P_{electric\ quadrupole}=\frac{\mu_0}{720\pi c^3}\sum_{ik}\dddot{Q}_{ik}\dddot{Q}_{ik}. \tag{7.84}$$

In orders of magnitude $Q\sim\left(\rho r_s^2\right)r_s^3\sim\rho r_s^5$, $\dddot{Q}\sim\rho r_s^5/t_s^3\sim\ddot{p}_s r_s/t_s\sim\ddot{p}_s v_s$ and therefore

$$\frac{P_{electric\ quadrupole}}{P_{electric\ dipole}}\sim\left(\frac{v_s}{c}\right)^2. \tag{7.85}$$

Thus, for slowing-moving distributions, the power emitted in electric-quadrupole radiation is smaller than the power emitted in electric-dipole radiation by a factor of order $(v/c)^2 \ll 1$. We see also that electric-quadrupole radiation is of the same order of magnitude as magnetic-dipole radiation.

Example: A simple radiating quadrupole source is an oscillating "spheroidal" distribution of charge such that **p** = 0 and

$$Q(t) = \begin{pmatrix} -\tfrac{1}{2}Q_0 & 0 & 0 \\ 0 & -\tfrac{1}{2}Q_0 & 0 \\ 0 & 0 & Q_0 \end{pmatrix} \cos\omega t \,. \tag{7.86}$$

The radiated power per solid angle is given by eq. (7.79) which is

$$\frac{dP}{d\Omega} = \frac{\mu_0}{576\pi^2 c^3}\left[\left|\dddot{\mathbf{Q}}(\hat{\mathbf{r}},t_0)\right|^2 - \left(\hat{\mathbf{r}}\cdot\dddot{\mathbf{Q}}(\hat{\mathbf{r}},t_0)\right)^2\right]. \tag{7.87}$$

In this case

$$\mathbf{Q}(\hat{\mathbf{r}},t_0) = \left\{-\tfrac{1}{2}Q_0\hat{x}, -\tfrac{1}{2}Q_0\hat{y}, Q_0\hat{z}\right\}\cos\omega t_0 = Q_0\left\{-\tfrac{1}{2}\sin\theta\cos\phi, -\tfrac{1}{2}\sin\theta\sin\phi, \cos\theta\right\}\cos\omega t_0 \,, \tag{7.88}$$

$$\dddot{\mathbf{Q}}(\hat{\mathbf{r}},t_0) = \omega^3 Q_0\left\{-\tfrac{1}{2}\sin\theta\cos\phi, -\tfrac{1}{2}\sin\theta\sin\phi, \cos\theta\right\}\sin\omega t_0 \,, \tag{7.89}$$

$$\left|\dddot{\mathbf{Q}}\right|^2 = \omega^6 Q_0^2\left[\frac{1}{4}\sin^2\theta + \cos^2\theta\right]\sin^2\omega t_0 \,, \tag{7.90}$$

$$\left(\hat{\mathbf{r}}\cdot\dddot{\mathbf{Q}}\right)^2 = \omega^6 Q_0^2\left(-\tfrac{1}{2}\sin^2\theta + \cos^2\theta\right)^2\sin^2\omega t_0 \,, \tag{7.91}$$

$$\left[\left|\dddot{\mathbf{Q}}\right|^2 - \left(\hat{\mathbf{r}}\cdot\dddot{\mathbf{Q}}\right)^2\right] = \frac{9}{4}\omega^6 Q_0^2\cos^2\theta\sin^2\theta\sin^2\omega t_0 \,. \tag{7.92}$$

The radiated power per solid angle is in this case

$$\frac{dP}{d\Omega} = \frac{\mu_0\omega^6 Q_0^2}{256\pi^2 c^3}\cos^2\theta\sin^2\theta\sin^2\omega t_0 \,. \tag{7.93}$$

After averaging over a complete wave cycle, this reduces to

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{\mu_0\omega^6 Q_0^2}{512\pi^2 c^3}\cos^2\theta\sin^2\theta \,. \tag{7.94}$$

This a four-lobed pattern with maxima at $\theta = \pi/4$ and $\theta = 3\pi/4$. The total radiated power by this quadrupole is

$$\langle P\rangle = \frac{\mu_0\omega^6 Q_0^2}{960\pi c^3} \,. \tag{7.95}$$

7.5 Power radiated by a point charge

In Section 6 (Eqs. 6.136) and (6.140), we derived the fields of a point charge q in arbitrary motion

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0}\frac{\xi}{(\boldsymbol{\xi}\cdot\mathbf{u})^3}\left[\left(c^2 - v^2\right)\mathbf{u} + \boldsymbol{\xi}\times\left(\mathbf{u}\times\mathbf{a}\right)\right], \tag{7.96}$$

where $\mathbf{u} \equiv c\hat{\boldsymbol{\xi}} - \mathbf{v}$ and $\boldsymbol{\xi} \equiv \mathbf{r} - \mathbf{w}(t_r)$, and

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{c}\hat{\boldsymbol{\xi}}\times\mathbf{E}(\mathbf{r},t) \,. \tag{7.97}$$

The first term in Eq. (7.96) is the *velocity field*, and the second one is the *acceleration field*.

Poynting's vector is

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} = \frac{1}{\mu_0 c}\mathbf{E}\times\left(\hat{\boldsymbol{\xi}}\times\mathbf{E}\right) = \left[\frac{1}{\mu_0 c}E^2\hat{\boldsymbol{\xi}} - \left(\hat{\boldsymbol{\xi}}\cdot\mathbf{E}\right)\mathbf{E}\right]. \tag{7.98}$$

However, not all of this energy flux constitutes *radiation*; some of it is just field energy carried along by the particle as it moves. The *radiated* energy *detaches* itself from the charge and propagates off to infinity. To calculate the total power *radiated* by the particle at time t_r, we consider a huge sphere of radius ξ (Fig. 7.2), centered at the position of the particle (at time t_r), wait the appropriate interval $t - t_r = \xi / c$ for the radiation to reach the sphere, and at that moment integrate Poynting's vector over the surface.

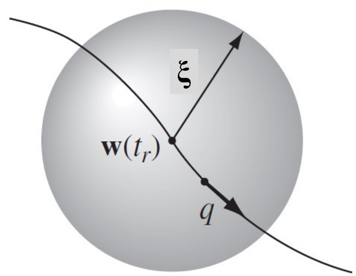

Fig. 7.2

Now, the area of the sphere is proportional to ξ^2, so any term in **S** that goes like $1/\xi^2$ will yield a finite result, but terms $\sim 1/\xi^3$ or $1/\xi^4$ will contribute nothing in the limit $\xi \to \infty$. For this reason, only the acceleration field represent true radiation and hence it is also known as *radiation field*:

$$\mathbf{E}_{rad} = \frac{q}{4\pi\varepsilon_0}\frac{\xi}{\left(\boldsymbol{\xi}\cdot\mathbf{u}\right)^3}\left[\boldsymbol{\xi}\times\left(\mathbf{u}\times\mathbf{a}\right)\right]. \tag{7.99}$$

Now $\mathbf{E}_{rad}$ is perpendicular to $\boldsymbol{\xi}$, so the second term in Eq. (7.98) vanishes:

$$\mathbf{S}_{rad} = \frac{1}{\mu_0 c}E_{rad}^2\,\hat{\boldsymbol{\xi}}. \tag{7.100}$$

If the charge is instantaneously at rest (at time t_r), then $\mathbf{u} = c\hat{\boldsymbol{\xi}}$, and

$$\mathbf{E}_{rad} = \frac{q}{4\pi\varepsilon_0}\frac{1}{\xi c^2}\left[\hat{\boldsymbol{\xi}}\times\left(\hat{\boldsymbol{\xi}}\times\mathbf{a}\right)\right] = \frac{q\mu_0}{4\pi\xi}\left[\left(\hat{\boldsymbol{\xi}}\cdot\mathbf{a}\right)\hat{\boldsymbol{\xi}} - \mathbf{a}\right]. \tag{7.101}$$

In that case

$$\mathbf{S}_{rad} = \frac{1}{\mu_0 c}E_{rad}^2\,\hat{\boldsymbol{\xi}} = \frac{1}{\mu_0 c}\left(\frac{q\mu_0}{4\pi\xi}\right)^2\left[a^2 - \left(\hat{\boldsymbol{\xi}}\cdot\mathbf{a}\right)^2\right]\hat{\boldsymbol{\xi}} = \frac{\mu_0 q^2 a^2}{16\pi c}\frac{\sin^2\theta}{\xi^2}\hat{\boldsymbol{\xi}}, \tag{7.102}$$

where θ is the angle between $\hat{\boldsymbol{\xi}}$ and $\mathbf{a}$. No power is radiated in the forward or backward direction—rather, it is emitted in a donut about the direction of instantaneous acceleration (Fig. 7.3).

The total power radiated is

$$P = \oint \mathbf{S}_{rad}\cdot\mathbf{n}da == \frac{\mu_0 q^2 a^2}{16\pi c}2\pi\int\frac{\sin^2\theta}{\xi^2}\xi^2\sin\theta d\theta = \frac{\mu_0 q^2 a^2}{6\pi c}. \tag{7.103}$$

Eq. (7.103) is known as the *Larmor formula*.

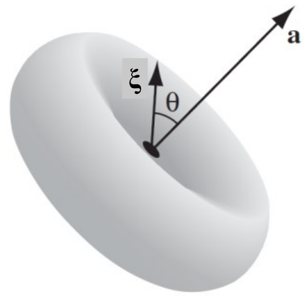

Fig. 7.3

The assumption that $v=0$, which was used to derive Eqs. (7.102) and (7.103), actually holds to good approximation as long as $v \ll c$. This because, in the context of special relativity, the condition $v=0$ simply represents an astute choice of reference system, with no essential loss of generality.

An exact treatment of the case $v \neq 0$ is harder, due to $\mathbf{E}_{rad}$ being more complicated and also due to $\mathbf{S}_{rad}$, the rate at which energy passes through the sphere, being not the same as the rate at which energy leaves the particle. Motion of the particle makes the energy rate dW/dt, at which energy passes through the sphere at radius ξ, not the same as the rate at which energy left the charge dW/dt_r:

$$\frac{dW}{dt_r} = \frac{dW/dt}{dt_r/dt} = \frac{1}{\dot{t}_r}\frac{dW}{dt}. \tag{7.104}$$

To find $\dot{t}_r$, we calculate derivative of ξ:

$$\frac{d\xi}{dt} = \frac{\partial}{\partial t}\sqrt{\boldsymbol{\xi}\cdot\boldsymbol{\xi}} = \frac{1}{2\xi}\frac{\partial}{\partial t}\left(\boldsymbol{\xi}\cdot\boldsymbol{\xi}\right) = \frac{1}{\xi}\frac{\partial \boldsymbol{\xi}}{\partial t}\cdot\boldsymbol{\xi} = -\frac{1}{\xi}\dot{t}_r\mathbf{v}\cdot\boldsymbol{\xi}. \tag{7.105}$$

On the other hand, by definition $t_r = t - \xi/c$ and hence $\frac{\partial \xi}{\partial t} = c\left(1-\dot{t}_r\right)$. Therefore, we obtain

$$-\frac{1}{\xi}\dot{t}_r\mathbf{v}\cdot\boldsymbol{\xi} = c\left(1-\dot{t}_r\right), \tag{7.106}$$

so that

$$\dot{t}_r = \frac{\xi c}{\xi c - \boldsymbol{\xi}\cdot\mathbf{v}} = \frac{\xi c}{\boldsymbol{\xi}\cdot\mathbf{u}}. \tag{7.107}$$

Coming back to Eq. (7.104), we have

$$\frac{dW}{dt_r} = \frac{\boldsymbol{\xi}\cdot\mathbf{u}}{\xi c}\frac{dW}{dt}. \tag{7.108}$$

It is easy to see that

$$\frac{\boldsymbol{\xi}\cdot\mathbf{u}}{\xi c} = 1 - \frac{\hat{\boldsymbol{\xi}}\cdot\mathbf{v}}{c}. \tag{7.109}$$

This represents a geometrical factor (the same as in the Doppler effect), reflecting a relative motion between the detector (the sphere to detect electromagnetic radiation) and the source (moving point charge). This factor needs to be used to correctly calculate the radiating power:

$$\frac{dP}{d\Omega} = \frac{\boldsymbol{\xi}\cdot\mathbf{u}}{\xi c}\frac{1}{\mu_0 c}E_{rad}^2\,\xi^2 = \frac{q^2}{16\pi^2\varepsilon_0}\frac{\left|\hat{\boldsymbol{\xi}}\times\left(\mathbf{u}\times\mathbf{a}\right)\right|^2}{\left(\hat{\boldsymbol{\xi}}\cdot\mathbf{u}\right)^5}, \tag{7.110}$$

where $d\Omega = \sin\theta d\theta d\phi$ is the solid angle into which this power is radiated. Integrating over θ and ϕ to get the total power radiated is not trivial, and for now we simply quote the answer:

$$P = \frac{\mu_0 q^2 \gamma^6}{6\pi c}\left(a^2 - \frac{|\mathbf{v}\times\mathbf{a}|^2}{c^2}\right), \tag{7.111}$$

where $\gamma \equiv 1/\sqrt{1-v^2/c^2}$. This is *Liénard's generalization* of the Larmor formula (to which it reduces when $v \ll c$). The factor γ^6 means that the radiated power increases enormously as the particle velocity approaches the speed of light.

Example: Suppose **v** and **a** are instantaneously collinear (at time t_r), in straight-line motion. We need to find the angular distribution of the radiation and the total power emitted.

In this case $(\mathbf{u}\times\mathbf{a}) = c(\hat{\boldsymbol{\xi}}\times\mathbf{a})$, so

$$\frac{dP}{d\Omega} = \frac{q^2 c^2}{16\pi^2\varepsilon_0}\frac{\left|\hat{\boldsymbol{\xi}}\times(\hat{\boldsymbol{\xi}}\times\mathbf{a})\right|^2}{\left(c-\hat{\boldsymbol{\xi}}\cdot\mathbf{v}\right)^5}. \tag{7.112}$$

Now

$$\hat{\boldsymbol{\xi}}\times(\hat{\boldsymbol{\xi}}\times\mathbf{a}) = (\hat{\boldsymbol{\xi}}\cdot\mathbf{a})\hat{\boldsymbol{\xi}} - \mathbf{a} \quad\Rightarrow\quad \left|\hat{\boldsymbol{\xi}}\times(\hat{\boldsymbol{\xi}}\times\mathbf{a})\right|^2 = a^2 - (\hat{\boldsymbol{\xi}}\cdot\mathbf{a})^2. \tag{7.113}$$

Assuming that that the particle moves along the z axis, i.e. $\hat{\mathbf{z}} \parallel \mathbf{v}$,we obtain

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}, \tag{7.114}$$

where $\beta \equiv v/c$. This is consistent, of course, with Eq. (7.102), in the case $v = 0$. However, for very large v ($\beta \approx 1$) the donut of radiation (Fig. 6.3) is stretched out and pushed forward by the factor $(1 - \beta\cos\theta)^{-5}$, as indicated in Fig. 7.4. Although there is still no radiation in precisely the forward direction, most of it is concentrated within an increasingly narrow cone about the forward direction.

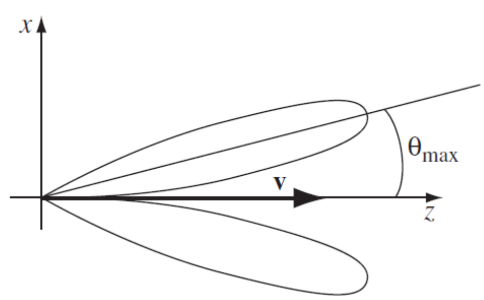

Fig. 7.4

The total power emitted is found by integrating Eq. (7.114) over all angles:

$$P = \int\frac{dP}{d\Omega}d\Omega = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\int\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}\sin\theta d\theta d\phi = \frac{\mu_0 q^2 a^2}{16\pi^2 c}2\pi\int_{-1}^{1}\frac{x^2}{(1-\beta x)^5}dx. \tag{7.115}$$

The integration yields $\frac{4}{3}(1-\beta^2)^{-3}$, and therefore

$$P = \frac{\mu_0 q^2 a^2 \gamma^6}{6\pi c} . \qquad (7.116)$$

This result is consistent with the Liénard formula (Eq. (7.111)), for the case of collinear **v** and **a**. Notice that the angular distribution of the radiation is the same whether the particle is accelerating or decelerating; it only depends on the square of *a*, and is concentrated in the forward direction (with respect to the velocity) in either case. When a high-speed electron hits a metal target it rapidly decelerates, giving off what is called *bremsstrahlung*, or "braking radiation."

7.6 Appendix

Here we prove the relationship

$$\mathbf{I} \equiv \int (\mathbf{e}\cdot\mathbf{r})\mathbf{J}(\mathbf{r},t) d^3r = \mathbf{m}(t)\times\mathbf{e} + \frac{1}{6}\dot{\mathbf{Q}}(\mathbf{e},t) + \frac{1}{6}\mathbf{e}\frac{d}{dt}\int \rho(\mathbf{r},t) r^2 d^3r , \qquad (7.117)$$

where **e** is arbitrary vector and

$$\mathbf{m}(t) = \frac{1}{2}\int_V \mathbf{r}\times\mathbf{J}(\mathbf{r},t) d^3r , \qquad (7.118)$$

$$Q_{ij}(t) = \int_V \rho(\mathbf{r},t)\left(3x_i x_j - r^2\delta_{ij}\right) d^3r , \qquad (7.119)$$

and vector $\mathbf{Q}(\mathbf{e},t)$ is defined as follows

$$Q_i(\mathbf{e},t) = \sum_j Q_{ij}(t) e_j . \qquad (7.120)$$

Consider *i* component of the integral (7.117)

$$I_i = \sum_j e_j \int x_j J_i(\mathbf{r},t) d^3r . \qquad (7.121)$$

For a localized current distribution

$$\int_V \nabla\left[x_i x_j \mathbf{J}\right] d^3r = \oint_S x_i x_j \mathbf{J}\cdot\mathbf{n} da = 0 , \qquad (7.122)$$

since there are now currents crossing surface *S*. On the other hand,

$$\int \nabla\left[x_i x_j \mathbf{J}(\mathbf{r})\right] d^3r = \int\left(x_j \mathbf{J}\cdot\nabla x_i + x_i \mathbf{J}\cdot\nabla x_j + x_i x_j \nabla\cdot\mathbf{J}\right) d^3r = \int\left(x_j J_i + x_i J_j - x_i x_j \frac{\partial\rho}{\partial t}\right) d^3r . \qquad (7.123)$$

Eqs. (7.122) and (7.123) yield

$$\frac{1}{2}\int x_j J_i d^3r = -\frac{1}{2}\int x_i J_j d^3r + \frac{1}{2}\frac{d}{dt}\int \rho x_i x_j d^3r . \qquad (7.124)$$

Eq. (7.124) allows us to write Eq. (7.121) in the form:

$$I_i = \sum_j e_j \int x_j J_i d^3r = -\frac{1}{2}\sum_j e_j \int\left(x_i J_j - x_j J_i\right) d^3r + \frac{1}{2}\frac{d}{dt}\sum_j e_j \int \rho x_i x_j d^3r . \qquad (7.125)$$

The term in brackets can be rewritten using the antisymmetric unit tensor ε_{ijk} :

$$\left(x_i J_j - x_j J_i\right) = \sum_k \varepsilon_{ijk}\left(\mathbf{r}\times\mathbf{J}\right)_k , \qquad (7.126)$$

so that

$$-\frac{1}{2}\sum_j e_j \int \left(x_i J_j - x_j J_i\right) d^3r = -\frac{1}{2}\int \sum_{jk} \varepsilon_{ijk} e_j \left(\mathbf{r}\times\mathbf{J}\right)_k d^3r = -\frac{1}{2}\int \left[\mathbf{e}\times\left(\mathbf{r}\times\mathbf{J}\right)\right]_i d^3r\,, \qquad (7.127)$$

The last term is related to quadrupole moment:

$$Q_i = \sum_j Q_{ij} e_j = \sum_j \int \rho\left(3x_i x_j - r^2\delta_{ij}\right) e_j d^3r = 3\sum_j e_j \int \rho x_i x_j d^3r - e_i \int \rho r^2 d^3r\ . \qquad (7.128)$$

Thus Eq. (7.125) can be written as follows:

$$I_i = \sum_j e_j \int x_j J_i d^3r = -\frac{1}{2}\int \left[\mathbf{e}\times\left(\mathbf{r}\times\mathbf{J}\right)\right]_i d^3r + \frac{\dot{Q}_i}{6} + \frac{e_i}{6}\frac{d}{dt}\int \rho r^2 d^3r = \left(\mathbf{m}\times\mathbf{e}\right)_i + \frac{\dot{Q}_i}{6} + \frac{e_i}{6}\frac{d}{dt}\int \rho r^2 d^3r\,. \qquad (7.129)$$