Section 7: Electrodynamics

Ohm's Law

We start the discussion of time-dependent electric and magnetic fields from revisiting Ohm's law. To generate an electric current flow, charges need to be affected by a force. The strength of the force determines how fast the charges move. For most substances, the current density **J** is proportional to the force per unit charge, **f**:

$$\mathbf{J} = \sigma \mathbf{f}\ . \tag{7.1}$$

The proportionality factor σ is called *conductivity* (not to be confused with surface charge) and depends on the nature of a material. The reciprocal of σ is called *resistivity*: $\rho = 1/\sigma$ (not to be confused with charge density). Notice that even insulators conduct slightly, though the conductivity of a metal is many orders of magnitude greater; in fact, for most purposes, metals can be regarded as perfect conductors, with $\sigma = \infty$, while for insulators, $\sigma = 0$.

For electromagnetic force $\mathbf{f} = \mathbf{E} + \mathbf{v} \times \mathbf{B}$, we obtain from Eq. (7.1)

$$\mathbf{J} = \sigma\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right). \tag{7.2}$$

Typically, the velocity of the charges is sufficiently small so that the second term can be ignored:

$$\mathbf{J} = \sigma \mathbf{E}\,. \tag{7.3}$$

Eq. (7.3) is called *Ohm's law*.

There is no contradiction with the known fact that **E** = 0 inside a conductor, because this assertion is valid for stationary charges where **J** = 0. Moreover, for perfect conductors $\mathbf{E} = \mathbf{J}/\sigma = 0$ even if current is flowing. In practice, metals are such good conductors that the electric field required to drive current in them is negligible. Thus, we routinely treat the connecting wires in electric circuits as equipotentials. Resistors, by contrast, are made from poorly conducting materials.

Example 1: A cylindrical resistor of cross-sectional area A and length L is made from material with conductivity σ (Fig. 7.1) The potential is constant over each end and the potential difference between the ends is V. We need to find current I flowing across the resistor.

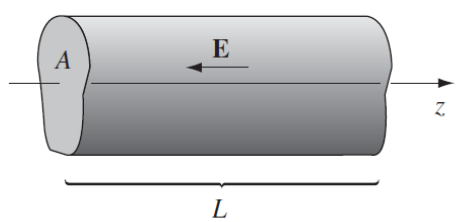


Fig. 7.1

The electric field is uniform within the wire and equal to $E = V/L$. The current is then

$$I = JA = \sigma EA = \frac{\sigma A}{L} V\ . \tag{7.4}$$

Example 2: Two long coaxial metal cylinders (radii a and b) are maintained at potential difference V and separated by material of conductivity σ (Fig. 7.2). We need to find current I flowing from one to the other, in a length L.

The field between the cylinders is

$$\mathbf{E} = \frac{\lambda}{2\pi\varepsilon_0 s}\hat{\mathbf{s}}\,, \tag{7.5}$$

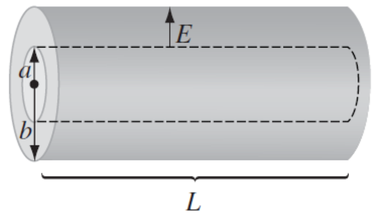


Fig. 7.2

where λ is the charge per unit length on the inner cylinder. The current is therefore

$$I = \int \mathbf{J} \cdot \mathbf{n} da = \sigma \int \mathbf{E} \cdot \hat{\mathbf{s}} da = \sigma E 2\pi s L = \frac{\sigma}{\varepsilon_0} \lambda L, \tag{7.6}$$

where the integral is over a cylindrical surface enclosing the inner cylinder. Meanwhile, the potential difference between the cylinders is

$$V = -\int_a^b \mathbf{E} \cdot d\mathbf{l} = \frac{\sigma}{2\pi\varepsilon_0} \ln\frac{b}{a}, \tag{7.7}$$

so

$$I = \frac{2\pi\sigma L}{\ln(b/a)} V. \tag{7.8}$$

In both examples, the total current flowing from one electrode to the other is proportional to the potential difference between them:

$$V = IR. \tag{7.9}$$

This is the more familiar version of Ohm's law. The constant of proportionality R is called the resistance. It depends on geometry of the resistor and conductivity of the medium between the electrodes. In Example 1, $R = \frac{L}{\sigma A}$ and in Example 2, $R = \frac{\ln(b/a)}{2\pi\sigma L}$. Resistance is measured in Ohms (Ω): $1\Omega = 1\frac{\text{V}}{\text{A}}$.

Note that for *steady* currents and *uniform* conductivity,

$$\nabla \cdot \mathbf{E} = \frac{1}{\sigma} \nabla \cdot \mathbf{J} = 0, \tag{7.10}$$

and therefore the charge density is zero; any unbalanced charge resides on the surface. It follows, in particular, that Laplace's equation holds within a homogeneous ohmic material carrying a steady current.

Ohm's law applies well to conductors. But, in fact, it may be surprising that Ohm's law ever holds. After all, a given field **E** produces a force $q\mathbf{E}$ on a charge q, and according to Newton's second law, the charge will accelerate. But if the charges are accelerating, why doesn't the current increase with time? Ohm's law implies, on the contrary, that a constant field produces a constant current, which suggests a constant velocity. Isn't that a contradiction to Newton's law?

This contradiction is lifted by the fact that the motion of charged particles that are accelerated by an electric field suffers energy and momentum changing collisions (known as scattering events) with other particles in the system. For example, in metals electrons suffer collisions with immobile ions. The linear dependence of **J** on **E** is one consequence of these collisions. Qualitatively, this can be seen if we balance the electric force on a conduction electron in a conductor with an effective drag force due to collisions. If τ is the average time between collisions known as the *relaxation time*, this balance makes it possible for the particles to achieve a terminal velocity **v** determined by

$$m\frac{d\mathbf{v}}{dt} = e\mathbf{E} - \frac{m\mathbf{v}}{\tau} = 0 . \qquad (7.11)$$

The steady solution of Eq. (7.11) is called the *drift velocity*.

$$\mathbf{v}_d = \frac{e\tau}{m}\mathbf{E} . \qquad (7.12)$$

The current density is determined by the drift velocity as follows:

$$\mathbf{J} = en_e\mathbf{v}_d , \qquad (7.13)$$

where n_e is the density of the charged particles (electrons) and e is their charge (positive because the current direction is determined by the flow of a positive charge). Substituting Eq. (7.13) into (7.12) gives Ohm's law with

$$\sigma = \frac{n_e e^2 \tau}{m} . \qquad (7.14)$$

This expression represents the classical *Drude model* for the electrical conductivity in a neutral system composed of free charges. It applies to a conductor in n_e mobile charged particles per unit volume of charge e, mass m, and relaxation time τ. The relaxation time can be determined empirically from experimental data on the conductivity. For copper, $n_e \simeq 8\times10^{28}$ free electrons per m^3 and at room temperature the conductivity is $\sigma \simeq 6\times10^{7}$ $\Omega^{-1}m^{-1}$. This gives $\tau \simeq 2\times10^{-14}$ s.

Electromotive Force

Now, we introduce the concept of *electromotive force*.

Consider a typical electric circuit. There are two forces involved in driving current around a circuit: the source, $\mathbf{f}_s$, which is ordinarily confined to one portion of the loop (a battery, say), and the electrostatic force, **E**, which serves to smooth out the flow and communicate the influence of the source to distant parts of the circuit. Therefore, the total force per unit charge in a circuit is

$$\mathbf{f} = \mathbf{f}_s + \mathbf{E} . \qquad (7.15)$$

The physical agency responsible for $\mathbf{f}_s$ can be different: in a battery it is a chemical force; in a piezoelectric crystal it is a mechanical pressure; in a thermocouple it is a temperature gradient; in a photoelectric cell it is light. Whatever the mechanism, its net effect is determined by the line integral of **f** around the circuit:

$$\mathcal{E} = \oint \mathbf{f}\cdot d\mathbf{l} = \oint \mathbf{f}_s \cdot d\mathbf{l} . \qquad (7.16)$$

The latter equality is because $\oint \mathbf{E}\cdot d\mathbf{l} = 0$ for electrostatic fields, and it doesn't matter whether you use **f** or $\mathbf{f}_s$. The quantity $\mathcal{E}$ is called the *electromotive force*, or *emf*, of the circuit. It is a lousy term, since this is not a force at all – it's the integral of a force per unit charge.

Within an *ideal* source of *emf* (a resistanceless battery, for instance), the net force on the charges is zero, so $\mathbf{E} = -\mathbf{f}_s$. The potential difference between the terminals (a and b) is therefore

$$\Delta\Phi = -\int_a^b \mathbf{E}\cdot d\mathbf{l} = \int_a^b \mathbf{f}_s \cdot d\mathbf{l} = \oint \mathbf{f}_s \cdot d\mathbf{l} = \mathcal{E} . \qquad (7.17)$$

We can extend the integral to the entire loop because $\mathbf{f}_s = 0$ outside the source. The function of a battery, then, is to establish and maintain a voltage difference equal to the electromotive force. The resulting electrostatic field drives current around the rest of the circuit (notice, however, that inside the battery $\mathbf{f}_s$ drives current in the direction opposite to **E**). Because it is the line integral of $\mathbf{f}_s$, $\mathcal{E}$ can be interpreted as the *work done, per unit charge*, by the source.

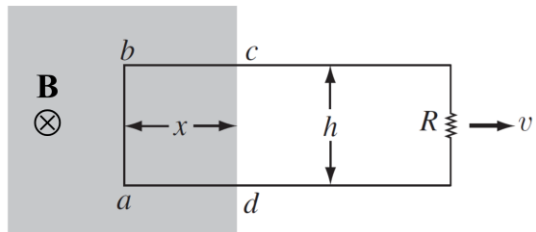


Fig. 7.3

One possible sources of electromotive force in a circuit is a *generator*. Generators exploit *motional emf*, which arises when you move a wire through a magnetic field. Figure 7.3 shows a primitive model for a generator. In the shaded region there is a uniform magnetic field **B**, pointing into the page, and the resistor *R* represents whatever it is we're trying to drive current through. If the entire loop is pulled to the right with speed v, the charges in segment experience a Lorentz force whose vertical component qvB drives current around the loop, in the clockwise direction. The *emf* is

$$\mathcal{E} = \oint \mathbf{f}_{mag} \cdot d\mathbf{l} = vBh\,. \tag{7.18}$$

There is a nice way to represent the *emf* generated in a moving loop. Let *F* be the flux of **B** through the loop:

$$F = \int \mathbf{B} \cdot \mathbf{n} da\,. \tag{7.19}$$

For the rectangular loop in Figure 7.3,

$$F = Bhx\,. \tag{7.20}$$

As the loop moves, the flux decreases

$$\frac{dF}{dt} = Bh\frac{dx}{dt} = -Bhv\,. \tag{7.21}$$

The minus sign accounts for the fact that dx/dt is negative. But this is precisely the *emf* given by Eq. (7.18); evidently the *emf* generated in the loop is minus the rate of change of flux through the loop:

$$\mathcal{E} = -\frac{dF}{dt}\,. \tag{7.22}$$

This is the *flux rule* for motional *emf*. Apart from its delightful simplicity, it has the virtue of applying to *nonrectangular* loops moving in *arbitrary* directions through *noniniform* magnetic fields; in fact, the loop need not even maintain a fixed shape. Now we prove this statement.

Figure 7.4 shows a loop of wire at time *t* and also a short time *dt* later. Suppose we compute the flux at time *t*, using surface *S*, and the flux at time *t* + *dt*, using the surface consisting of *S* plus the "ribbon" that connects the new position of the loop to the old. The change in flux, then, is

$$dF = F(t+dt) - F(t) = \int_{ribbon} \mathbf{B} \cdot \mathbf{n} da\,. \tag{7.23}$$

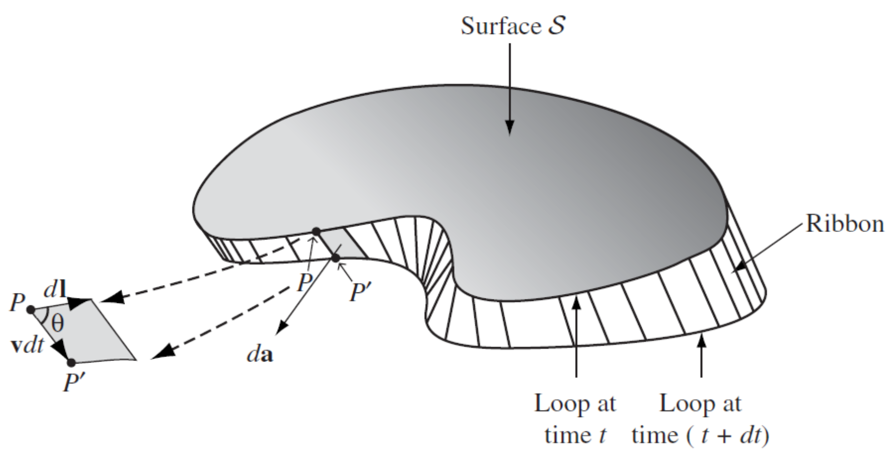


Fig. 7.4

Focus our attention on point P: in time dt it moves to P'. Let **v** be the velocity of the wire, and **u** the velocity of a charge down the wire; **w** = **v** + **u** is the resultant velocity of a charge at P. The infinitesimal element of area on the ribbon can be written as

$$\mathbf{n}da = (\mathbf{v} \times d\mathbf{l})dt \; . \tag{7.24}$$

(see inset in Fig. 7.4). Therefore

$$\frac{dF}{dt} = \oint \mathbf{B} \cdot (\mathbf{v} \times d\mathbf{l}) \, . \tag{7.25}$$

Since **w** = **v** + **u** and **u** is parallel to $d\mathbf{l}$, we can also write this as

$$\frac{dF}{dt} = \oint \mathbf{B} \cdot (\mathbf{w} \times d\mathbf{l}) = -\oint (\mathbf{w} \times \mathbf{B}) \cdot d\mathbf{l} \, . \tag{7.26}$$

But ($\mathbf{w} \times \mathbf{B}$) is the Lorentz force per unit charge, $\mathbf{f}_{mag}$, so

$$\frac{dF}{dt} = -\oint \mathbf{f}_{mag} \cdot d\mathbf{l} \tag{7.27}$$

and the integral of $\mathbf{f}_{\text{mag}}$ is the *emf*

$$\mathcal{E} = -\frac{dF}{dt} \, . \tag{7.28}$$

Faraday's Law

The first quantitative observations relating time-dependent electric and magnetic fields were made by Faraday (in 1831) in experiments on the behavior of currents in circuits placed in time-varying magnetic fields. The three of these experiments (with some violence to history) can be characterized as follows. A transient current is induced in a circuit if

(a) the circuit is moved through a magnetic field (Fig. 7.5a);

(b) a magnetic field is moved into or out of the circuit (Fig. 7.5b);

(c) the strength of the magnetic field is changed (Fig. 7.5c).

The first experiment, of course, is an example of motional *emf*, conveniently expressed by the flux rule (7.28). It is not surprising that exactly the same *emf* arises in Experiment 2 – all that really matters is the *relative* motion of the magnet and the loop. Indeed, in the light of special relativity it *has* to be so. But Faraday knew nothing of relativity, and in classical electrodynamics this simple reciprocity has remarkable implications. For the *moving* loop, it's a *magnetic* force that sets up the *emf*, but if the *stationary* loop, the force *cannot* be magnetic – stationary charges experience no magnetic forces. *Electric* fields could produce the force, and Faraday made an ingenious prediction: *A changing magnetic field induces an electric field*.

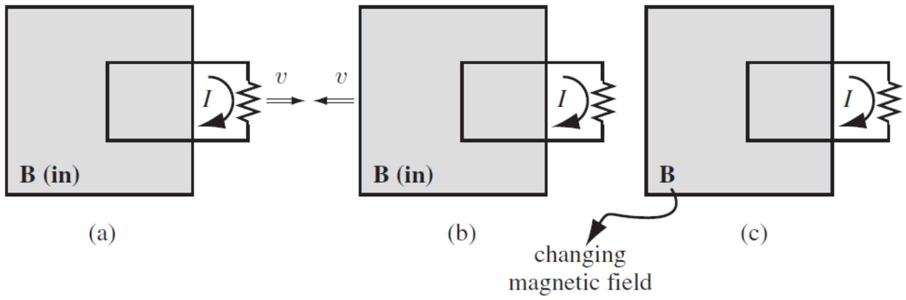


Fig. 7.5

Faraday attributed the transient current flow to a changing magnetic flux linked by the circuit. The changing flux induces an electric field around the circuit, the line integral of which is called the electromotive force,

$$\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} \,. \tag{7.29}$$

The electromotive force causes a current flow, according to Ohm's law. All the three experiments can therefore be described by Eq. (7.28) which takes the form

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{dF}{dt} \,. \tag{7.30}$$

and is called the *Faraday's law*. The negative sign in this equation is consistent with Lenz's law. According to Lenz's law, the current induced around a closed loop is always such that the magnetic field it produces tries to counteract the change in magnetic flux which generates the electromotive force.

Now let us consider the connection between Experiment 1 and Experiment 2 in a more detail. We know that when we are dealing with relative speeds that are small compared with the velocity of light, physical laws should be invariant under Galilean transformations. That is, physical phenomena are the same when viewed by two observers moving with a constant velocity $\mathbf{v}$ relative to one another, provided the coordinates in space and time are related by the Galilean transformation, $\mathbf{r}' = \mathbf{r} - \mathbf{v}t$, $t' = t$. Faraday's observations suggest that the same current is induced in a circuit whether it is moved while the magnetic field is stationary (Experiment 1) or it is held fixed while the magnetic field is moved into or out of the circuit in the same relative manner (Experiment 2).

Let us now consider Faraday's law for a moving circuit and see the consequences of Galilean invariance. Expressing (7.30) in terms of the integrals over $\mathbf{E}'$ and $\mathbf{B}$, we have

$$\oint_C \mathbf{E}' \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} da \,. \tag{7.31}$$

The induced electromotive force is proportional to the *total* time derivative of the flux – the flux can be changed by changing the magnetic induction or by changing the shape or orientation or position of the circuit. In form (7.31) we have a far-reaching generalization of Faraday's law. The circuit C can be thought of as any closed geometrical path in space, not necessarily coincident with an electric circuit. Then (7.31) becomes a relation between the fields themselves. It is important to note, however, that the electric field,

$\mathbf{E}'$ is the electric field at $d\mathbf{l}$ in the moving coordinate system in which $d\mathbf{l}$ is at rest, since it is that field that causes current to flow if a circuit is actually present.

If the circuit C is moving with a velocity $\mathbf{v}$ in some direction, the total time derivative in (7.31) must take into account this motion. The flux through the circuit may change because (a) the flux changes with time at a point, or (b) the translation of the circuit changes the location of the boundary. It is easy to show that the result for the total time derivative of flux through the moving circuit is

$$\frac{d}{dt}\int_S \mathbf{B}\cdot\mathbf{n}da = \int_S \frac{\partial \mathbf{B}}{\partial t}\cdot\mathbf{n}da + \oint_C (\mathbf{B}\times\mathbf{v})\cdot d\mathbf{l}. \tag{7.32}$$

Indeed, taking into account that the *convective* derivative is given by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla, \tag{7.33}$$

we can write

$$\frac{d\mathbf{B}}{dt} = \frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{B} = \frac{\partial \mathbf{B}}{\partial t} + \nabla\times(\mathbf{B}\times\mathbf{v}) + \mathbf{v}(\nabla\cdot\mathbf{B}), \tag{7.34}$$

where $\mathbf{v}$ is treated as a *fixed* vector in the differentiation. The last term is zero due to $\nabla\cdot\mathbf{B}=0$. Use of Stokes's theorem on the second term yields (7.32).

Equation (7.31) can now be written in the form,

$$\oint_C \left[\mathbf{E}' - (\mathbf{v}\times\mathbf{B})\right]\cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot\mathbf{n}da. \tag{7.35}$$

This is an equivalent statement of Faraday's law applied to the moving circuit C. But we can choose to interpret it differently. We can think of the circuit C and surface S as instantaneously at a certain position in space in the laboratory. Applying Faraday's law (7.31) to that fixed circuit, we find

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}\cdot\mathbf{n}da = -\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot\mathbf{n}da. \tag{7.36}$$

where $\mathbf{E}$ is now the electric field in the laboratory. The assumption of Galilean invariance implies that the left-hand sides of (7.35) and (7.36) must be equal. This means that the electric field $\mathbf{E}'$ in the moving coordinate system of the circuit is

$$\mathbf{E}' = \mathbf{E} + (\mathbf{v}\times\mathbf{B}). \tag{7.37}$$

Because we considered a Galilean transformation, the result (7.37) is an approximation valid only for speeds small compared to the speed of light. Faraday's law is no approximation, however.

Faraday's law can be put in differential form by use of Stokes's theorem. The transformation of the electromotive force integral into a surface integral leads to

$$\int \left(\nabla\times\mathbf{E} + \frac{\partial \mathbf{B}}{\partial t}\right)\cdot\mathbf{n}da = 0. \tag{7.38}$$

Since the circuit C and bounding surface S are arbitrary, the integrand must vanish at all points in space. Thus the differential form of Faraday's law is

$$\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{7.39}$$

This is the time-dependent generalization of the statement, $\nabla\times\mathbf{E}=0$, for electrostatic fields.

Magnetostatic Energy

In discussing steady-state magnetic fields, we avoided the question of field energy and energy density. The reason was that the creation of a steady-state configuration of currents and associated magnetic fields involves an initial transient period during which the currents and fields are brought from zero to the final values. For such time-varying fields there are induced electromotive forces that cause the sources of current to do work. Since the energy in the field is by definition the total work done to establish it, we must consider these contributions.

Suppose for a moment that we have only a single circuit with a constant current I flowing in it. If the flux through the circuit changes, an electromotive force $\mathcal{E}$ is induced around it. To keep the current constant, the sources of current must do work. The work done on a unit positive charge, against the *emf*, in one trip around a circuit is $-\mathcal{E}$ (the minus sign records the fact that this is the work done against the *emf*, not the work done by *emf*). The amount of charge per unit time passing down the wire is I. So, the total work done per unit time is

$$\frac{dW}{dt} = -I\mathcal{E} = I\frac{dF}{dt}. \tag{7.40}$$

This is in addition to ohmic losses in the circuit, which are not to be included in the magnetic energy. Thus, if the flux change through a circuit carrying a current I is δF, the work done by the sources is

$$\delta W = I\delta F . \tag{7.41}$$

We can express the increment of work done against the induced *emf* in terms of the change in magnetic field through the loop:

$$\delta F = \delta\int_S \mathbf{B}\cdot\mathbf{n}da = \delta\int_S (\nabla\times\mathbf{A})\cdot\mathbf{n}da = \delta\oint_C \mathbf{A}\cdot d\mathbf{l} = \oint_C \delta\mathbf{A}\cdot d\mathbf{l}, \tag{7.42}$$

where C is the curve bounding surface S. Thus,

$$\delta W = I\oint_C \delta\mathbf{A}\cdot d\mathbf{l} = \oint_C (\delta\mathbf{A}\cdot\mathbf{I})dl . \tag{7.43}$$

In this form the generalization to the volume currents is obvious:

$$\delta W = \int_V (\delta\mathbf{A}\cdot\mathbf{J})d^3r . \tag{7.44}$$

An expression involving the magnetic fields rather than $\mathbf{J}$ and $\delta\mathbf{A}$ can be obtained by using the Ampere's law $\nabla\times\mathbf{H} = \mathbf{J}$. Then

$$\delta W = \int_V \delta\mathbf{A}\cdot(\nabla\times\mathbf{H})d^3r . \tag{7.45}$$

Using the vector identity $\nabla\cdot(\mathbf{v}\times\mathbf{w}) = \mathbf{w}\cdot(\nabla\times\mathbf{v}) - \mathbf{v}\cdot(\nabla\times\mathbf{w})$, Eq. (7.45) can be transformed to

$$\delta W = \int_V \left[\mathbf{H}\cdot(\nabla\times\delta\mathbf{A}) + \nabla\cdot(\mathbf{H}\times\delta\mathbf{A})\right]d^3r . \tag{7.46}$$

If the current distribution is assumed to be localized and we integrate over all space, the second integral vanishes. With the definition of $\mathbf{B}$ in terms of $\mathbf{A}$, the energy increment can be written:

$$\delta W = \int \mathbf{H}\cdot\delta\mathbf{B}d^3r . \tag{7.47}$$

This relation is the magnetic equivalent of the electrostatic equation (4.87). In its present form it is applicable to all magnetic media, including ferromagnetic substances. If we assume that the medium is para- or diamagnetic, so that a linear relation exists between **H** and **B** (i.e. $\mathbf{B} = \mu\mathbf{H}$), then

$$\mathbf{H}\cdot\delta\mathbf{B} = \frac{1}{2}\delta\left(\mathbf{H}\cdot\mathbf{B}\right). \tag{7.48}$$

If we now bring the fields up from zero to their final values, the total magnetic energy will be

$$W = \frac{1}{2}\int \mathbf{H}\cdot\mathbf{B}d^3r \ . \tag{7.49}$$

In view of this result, we can say that the energy is "stored in the magnetic field", in the amount of $\frac{1}{2}\mathbf{H}\cdot\mathbf{B}$ per unit volume.

Eq. (7.49) is the magnetic analog of the respective electrostatic equation. The magnetic equivalent of the equation where the electrostatic energy is expressed in terms of the charge density and the potential can be obtained from Eq. (7.44) by assuming linear relation between **J** and **A**. Then we find the magnetic energy to be

$$W = \frac{1}{2}\int\left(\mathbf{A}\cdot\mathbf{J}\right)d^3r \ . \tag{7.50}$$

Self- and Mutual Inductances

The concept of self- and mutual inductances is useful for systems of current-carrying circuits. Imagine a system of N distinct current-carrying circuits, with I_i being the total current carrying by the i-th circuit. The circuits are not necessarily thin wires but are assumed for the present to be non-permeable. It appears that the total energy of the system of currents (7.50) can be expressed as

$$W = \frac{1}{2}\sum_{i=1}^{N} L_i I_i^2 + \sum_{i=1}^{N}\sum_{j>i}^{N} M_{ij} I_i I_j \ . \tag{7.51}$$

where L_i is the self-inductance of the i-th circuit and M_{ij} is the mutual inductance between the i-th and j-th circuits. To establish this result, we first use the expression for the vector potential

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r' . \tag{7.52}$$

to convert (7.50) to

$$W = \frac{\mu_0}{8\pi}\int d^3r\int d^3r'\frac{\mathbf{J}(\mathbf{r})\cdot\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \ . \tag{7.53}$$

The integrals can now be broken up into sums of separate integrals over each circuit:

$$W = \frac{\mu_0}{8\pi}\sum_{i=1}^{N}\int_{C_i} d^3r_i\sum_{j=1}^{N}\int_{C_j} d^3r_j'\frac{\mathbf{J}(\mathbf{r}_i)\cdot\mathbf{J}(\mathbf{r}_j')}{|\mathbf{r}_i-\mathbf{r}_j'|} \ . \tag{7.54}$$

In the sums there are terms with $i = j$ and terms with $i \neq j$. The former defines the first sum in Eq. (7.51) and the latter, the second. Evidently, the coefficients L_i and M_{ij} are given by

$$L_i = \frac{\mu_0}{4\pi I_i^2}\int_{C_i} d^3r_i\int_{C_i} d^3r_i'\frac{\mathbf{J}(\mathbf{r}_i)\cdot\mathbf{J}(\mathbf{r}_i')}{|\mathbf{r}_i-\mathbf{r}_i'|} \ , \tag{7.55}$$

$$M_{ij} = \frac{\mu_0}{4\pi I_i I_j}\int_{C_i} d^3r_i\int_{C_j} d^3r_j'\frac{\mathbf{J}(\mathbf{r}_i)\cdot\mathbf{J}(\mathbf{r}_j')}{|\mathbf{r}_i-\mathbf{r}_j'|} \ . \tag{7.56}$$

Note that the coefficients of mutual inductance M_{ij} are symmetric in i and j.

These general expressions for self- and mutual inductance are the rigorous versions of the more elementary definitions in terms of flux linkage. To establish the connection, consider the expression for mutual inductance (for which the ambiguities in the definition of flux linkage for self-inductance are absent). The integral over d^3r_j' times $\mu_0/4\pi$ is just the expression (7.52) for the vector potential $\mathbf{A}_j(\mathbf{r}_i)$ at position $\mathbf{r}_i$ in the i-th circuit caused by the current I_j, flowing in the j-th circuit:

$$\mathbf{A}_j(\mathbf{r}_i) = \frac{\mu_0}{4\pi}\int_{C_j} d^3r_j' \frac{\mathbf{J}(\mathbf{r}_j')}{\left|\mathbf{r}_i - \mathbf{r}_j'\right|} \quad . \tag{7.57}$$

So that Eq. (7.56) can be written in form

$$M_{ij} = \frac{1}{I_i I_j}\int_{C_i} \mathbf{A}_j(\mathbf{r}_i)\cdot\mathbf{J}(\mathbf{r}_i)d^3r_i \quad . \tag{7.58}$$

If the i-th circuit is imagined to be negligible in cross section compared to the overall scale of both circuits, we can write the integrand $\mathbf{J}(\mathbf{r}_i)d^3r_i$ for the integration over the volume of the i-th circuit as $\mathbf{J}d^3r = J_{\|}da d\mathbf{l}$, where da is a locally defined element of cross-sectional area and $d\mathbf{l}$ is a directed longitudinal differential in the sense of current flow. With the vector potential sensibly constant in the cross-sectional integral at a fixed position along the circuit, the mutual inductance becomes

$$M_{ij} = \frac{1}{I_i I_j}\int_{C_i} \mathbf{A}_{ij}\cdot J_{\|}da d\mathbf{l} = \frac{1}{I_i I_j}\oint_{C_i} \mathbf{A}_{ij}\cdot d\mathbf{l}\int_{S_i} J_{\|}da = \frac{1}{I_j}\oint_{C_i} \mathbf{A}_{ij}\cdot d\mathbf{l} = \frac{1}{I_j}\int_{S_i}\left(\nabla\times\mathbf{A}_{ij}\right)\cdot\mathbf{n}da \quad . \tag{7.59}$$

where $\mathbf{A}_{ij}$ is the vector potential caused by the j-th circuit at the integration point on the i-th and the factor I_i, comes from the integral over the cross section $I_i = \int_{S_i} J_{\|}da$. Stokes's theorem has been used to obtain the second form. Since the curl of $\mathbf{A}$ is the magnetic field $\mathbf{B}$, the area integral is just the magnetic-flux linkage

$$F_{ij} = \int_{S_i} \mathbf{B}_{ij}\cdot\mathbf{n}da \quad . \tag{7.60}$$

Thus, the mutual inductance is finally

$$M_{ij} = \frac{1}{I_j}F_{ij}, \tag{7.61}$$

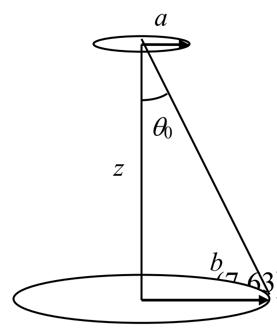
where F_{ij} is the magnetic flux from circuit j linked within circuit i. For self-inductance, the physical argument is the same, but the ambiguity in the meaning of the self-flux linkage F_{ii} requires a return to the rigorous expression (7.55) based on the magnetic energy.

Another way to represent mutual inductance from linear current circuits is to use Eq. (7.56) directly. Similar to the arguments used in the derivation of Eq. (7.59), we can integrate over the cross-sectional area of the circuits and write

$$M_{ij} = \frac{\mu_0}{4\pi}\int_{C_i}\int_{C_j}\frac{d\mathbf{l}_i\cdot d\mathbf{l}_j}{\left|\mathbf{r}_i - \mathbf{r}_j\right|} \quad . \tag{7.62}$$

This is the *Neumann formula*; it involves a double line integral – one integration around loop i and the other around loop j. It is not very useful for practical calculations, but it reveals that mutual inductance is a purely geometrical quantity, which depends only on the size, shape, and relative position of the two loops.

Example: A small loop of wire of radius a lies at distance z above the center of a large loop of radius b, as shown in the figure. The planes of the two loops are parallel, and perpendicular to the common axis. Find the mutual inductances and confirm that $M_{12} = M_{21}$.



We use formula (7.61) to calculate the mutual inductance. The z-component of the field produced by the loop of radius b is given by

$$B_z = \frac{\mu_0 I}{4\pi}\int_{C_b}\frac{\sin\theta_0 dl}{b^2+z^2} = \frac{\mu_0 I}{2}\frac{b^2}{\left(b^2+z^2\right)^{3/2}} \ , \tag{7.63}$$

where we took into account that $\sin\theta_0 = b/\sqrt{b^2+z^2}$. Hence, the flux of the magnetic field crossing the area of loop a is

$$F = \int_{S_a}\mathbf{B}\cdot\mathbf{n}da = \frac{\mu_0 I}{2}\frac{\pi a^2 b^2}{\left(b^2+z^2\right)^{3/2}} \ . \tag{7.64}$$

Similarly, we can calculate the flux of a magnetic field produced by loop a and crossing the area of loop b. The magnetic field can be evaluated within the dipole approximation because the loop a is assumed to be small. In this case, in the spherical coordinates, we have

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4}\frac{a^2}{r^3}\left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}\right) \ . \tag{7.65}$$

Taking into account that the flux of magnetic field in independent of the choice of the bounding surface and by integrating over a spherical surface bounded by loop b and centered at loop a, we obtain:

$$F = \int_{S_b}\mathbf{B}\cdot\hat{\mathbf{r}}da = \frac{\mu_0 I}{4\pi}\frac{a^2}{r^3}\int 2\cos\theta r^2\sin\theta d\theta d\phi = \frac{\mu_0 I}{4\pi}\frac{a^2}{r}4\pi\int_0^{\theta_0}\cos\theta\sin\theta d\theta = \frac{\mu_0 I}{2}\frac{\pi a^2 b^2}{\left(b^2+z^2\right)^{3/2}} \ , \tag{7.66}$$

where we took into account that $r = \sqrt{b^2+z^2}$ and $\sin\theta_0 = b/r$. The result (7.66) is identical to (7.64). Therefore the mutual inductance is

$$M_{12} = M_{21} = \frac{\mu_0 \pi a^2 b^2}{2\left(b^2+z^2\right)^{3/2}} \ . \tag{7.67}$$

Maxwell's Equations

All the electromagnetism laws discussed in preceding sections can be summarized in four equations

Coulomb's law (Gauss's law): $$\nabla\cdot\mathbf{D} = \rho , \tag{7.68}$$

Ampere's law: $$\nabla\times\mathbf{H} = \mathbf{J} , \tag{7.69}$$

Absence of free magnetic poles: $$\nabla\cdot\mathbf{B} = 0 , \tag{7.70}$$

Faraday's law: $$\nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0 . \tag{7.71}$$

These equations represent the state of electromagnetic theory before Maxwell. It appears that there is a fatal inconsistency in these equations. All the equations except the Faraday's law were derived from steady-state observations. However, there is no *a priopi* reason to expect that the static equations will hold unchanged for time dependent fields.

The inconsistency has to do with the rule that divergence of curl is always zero. If you apply the divergence to Eq. (7.71), everything works out:

$$\nabla\cdot(\nabla\times\mathbf{E}) = -\nabla\cdot\frac{\partial\mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla\cdot\mathbf{B}) = 0. \quad (7.72)$$

The left side is zero because divergence of curl is zero; the right side is zero by virtue of equation (7.70). But when we do the same thing to Eq. (7.69), we get into trouble:

$$\nabla\cdot(\nabla\times\mathbf{H}) = \nabla\cdot\mathbf{J}, \quad (7.73)$$

the left side must be zero, but the right side, in general, is not. For steady currents, the divergence of **J** is zero, but evidently when we go beyond magnetostatics Ampere’s law cannot be right.

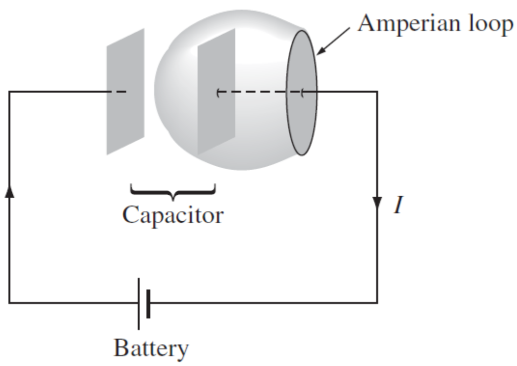


Fig. 7.6

There's another way to see that Ampere’s law fails for nonsteady currents. Suppose we are in the process of charging up a capacitor (Fig. 7.6). In integral form, Ampere’s law reads

$$\oint \mathbf{H}\cdot d\mathbf{l} = I_{encl}, \quad (7.74)$$

where I_{encl} is the total current enclosed by the loop. If we apply it to the amperian loop shown in the diagram the result appears to be dependent on the choice of the surface bounded by the loop. For the surface lying in the plane of the loop – the wire punctures this surface, so the enclosed current is $I_{encl} = I$. However, for the balloon-shaped surface in Fig. 7.6, no current passes through this surface, and we conclude that $I_{encl} = 0$! We never had this problem in magnetostatics because the conflict arises only when charge is piling up somewhere (in this case, on the capacitor plates). But for non-steady currents (such as this one) "the current enclosed by a loop" is an ill-defined notion, since it depends entirely on what surface you use.

Of course, we had no right to expect Ampere's law to hold outside of magnetostatics; after all, we derived it from the Biot-Savart law. However, in Maxwell's time there was no experimental reason to doubt that Ampere's law was of wider validity. The flaw was a purely theoretical one, and Maxwell fixed it by purely theoretical arguments.

The problem is on the right side of Eq. (7.73), which should be zero, but isn't. Applying the continuity equation and Gauss's law, the offending term can be rewritten:

$$\nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial t}(\nabla\cdot\mathbf{D}) = -\nabla\cdot\frac{\partial\mathbf{D}}{\partial t}. \quad (7.75)$$

It occurs that if we were to combine $\frac{\partial \mathbf{D}}{\partial t}$ with **J**, in Ampere's law, it would be just right to kill off the extra divergence.

New formulation of Ampere's law: $$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}. \quad (7.76)$$

Such a modification changes nothing, as far as magnetostatics is concerned: when **D** is constant, we still have $\nabla \times \mathbf{H} = \mathbf{J}$. In fact, Maxwell's term is hard to detect in ordinary electromagnetic experiments, that's why Faraday and the others never discovered it in the laboratory. However, it plays a crucial role in the propagation of electromagnetic waves, as we'll see below. Without it there would be no electromagnetic radiation. It was Maxwell's prediction that light is an electromagnetic wave phenomenon.

Maxwell's Eq. (7.76) suggests that just as a changing magnetic field induces an electric field (Faraday's law), *a changing electric field induces a magnetic field*. Of course, theoretical convenience is only suggestive – there might, after all, be other ways to fix Ampere's law. The real confirmation of Maxwell's theory came in 1888 with Hertz's experiments on electromagnetic waves.

Maxwell called his extra term the displacement current:

$$\mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t}. \quad (7.77)$$

Let's see now how the displacement current resolves the paradox of the charging capacitor (Fig. 3.3). If the capacitor plates are very close together, then the electric displacement field between them is

$$D = \sigma = \frac{Q}{A}, \quad (7.78)$$

where Q is the charge on the plate and A is its area. Thus, between the plates

$$\frac{\partial D}{\partial t} = \frac{dQ}{A\partial t} = \frac{I}{A}. \quad (7.79)$$

Now, Eq. (7.76) reads, in integral form,

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{encl} + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot \mathbf{n} da. \quad (7.80)$$

If we choose the flat surface, then $D = 0$ and $I_{encl} = I$. If, on the other hand, we use the balloon-shaped surface, then $I_{encl} = 0$, but $\oint \frac{\partial \mathbf{D}}{\partial t} \cdot \mathbf{n} da = I$. So, we get the same answer for either surface, though in the first case it comes from the genuine current and in the second from the displacement current.

The final set of Maxwell equations can be written as follows

Coulomb's law (Gauss law): $$\nabla \cdot \mathbf{D} = \rho, \quad (7.81)$$

Ampere's law: $$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \quad (7.82)$$

Absence of free magnetic poles: $$\nabla \cdot \mathbf{B} = 0, \quad (7.83)$$

Faraday's law: $$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0. \quad (7.84)$$

These equations combined with constitutive relations connecting **E** and **B** with **D** and **H** form the basis of all classical electrodynamics.