Section 6: Potentials and Fields

6.1 Potential formulation of Maxwell's equations

Now we consider a *general* solution of Maxwell's equations. Namely we are interested in how the sources (charges and currents) generate electric and magnetic fields. For simplicity we restrict our considerations to the vacuum. In this case, Maxwell's equations have the form:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \tag{6.1}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{6.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{6.3}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}. \tag{6.4}$$

Maxwell's equations consist of a set of coupled first-order partial differential equations relating the various components of electric and magnetic fields. They can be solved as they stand in simple situations. But it is often convenient to introduce potentials, obtaining a smaller number of second-order equations, while satisfying some of Maxwell's equations identically. We are already familiar with this concept in electrostatics and magnetostatics, where we used the scalar potential Φ and the vector potential **A**.

Since $\nabla \cdot \mathbf{B} = 0$ still holds, we can define **B** in terms of a vector potential:

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{6.5}$$

Then the Faraday's law (6.3) can be written

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0. \tag{6.6}$$

This means that the quantity with vanishing curl in Eq. (6.6) can be written as the gradient of some scalar function, namely, a scalar potential Φ:

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi, \tag{6.7}$$

or

$$\mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}. \tag{6.8}$$

The definition of **B** and **E** in terms of the potentials **A** and Φ according to Eqs. (6.5) and (6.8) satisfies identically the two homogeneous Maxwell's equations (6.2) and (6.3). The dynamic behavior of **A** and Φ are determined by the two inhomogeneous Eqs. (6.1) and (6.4). Putting Eq. (6.8) into (6.1), we find that

$$\nabla^2 \Phi + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\varepsilon_0}. \tag{6.9}$$

This equation replaces Poisson's equation (to which it reduces in the static case). Substituting Eqs. (6.5) and (6.8) into Eq. (6.4) yields

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J} - \frac{1}{c^2}\nabla\left(\frac{\partial \Phi}{\partial t}\right) - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2}. \tag{6.10}$$

Now using the vector identity $\nabla\times(\nabla\times\mathbf{A})=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}$ and rearranging the terms, we find

$$\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}=\mu_0\mathbf{J}-\frac{1}{c^2}\nabla\left(\frac{\partial\Phi}{\partial t}\right)-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}, \tag{6.11}$$

or

$$\nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}-\nabla\left(\nabla\cdot\mathbf{A}+\frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right)=-\mu_0\mathbf{J}\ . \tag{6.12}$$

We have now reduced the set of four Maxwell's equations to two Eqs. (6.9) and (6.12). But they are still coupled equations. The uncoupling can be accomplished by exploiting the arbitrariness involved in the definition of the potentials. Since **B** is defined through Eq. (6.5) in terms of **A**, the vector potential is arbitrary to the extent that the gradient of some scalar function **A** can be added. Thus, **B** is left unchanged by the transformation,

$$\mathbf{A}\to\mathbf{A}'=\mathbf{A}+\nabla\Lambda\ . \tag{6.13}$$

For the electric field (6.8) to be unchanged as well, the scalar potential must be simultaneously transformed, as follows

$$\Phi\to\Phi'=\Phi-\frac{\partial\Lambda}{\partial t}\ . \tag{6.14}$$

The freedom implied by Eqs. (6.13) and (6.14) means that we can choose a set of potentials (**A**, Φ) such that

$$\nabla\cdot\mathbf{A}+\frac{1}{c^2}\frac{\partial\Phi}{\partial t}=0\ . \tag{6.15}$$

This uncouples the pair of equations (6.9) and (6.12) and leave two inhomogeneous wave equations, one for Φ and one for **A**:

$$\nabla^2\Phi-\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}=-\frac{\rho}{\varepsilon_0}, \tag{6.16}$$

$$\nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=-\mu_0\mathbf{J}\ . \tag{6.17}$$

Eqs. (6.16) and (6.17), plus (6.15), are equivalent in all respects to Maxwell's equations.

Example: Find charge and current distributions that would give rise to the potentials

$$\Phi=0\ , \tag{6.18}$$

$$\mathbf{A}=\begin{cases}\dfrac{\mu_0 k}{4c}\left(ct-|x|\right)^2\hat{\mathbf{z}}, & |x|<ct,\\ 0, & |x|>ct,\end{cases} \tag{6.19}$$

where *k* is a constant and *c* is the speed of light.

First, we determine the electric and magnetic fields, using Eqs. (6.8) and (6.5), we find for $|x|<ct$:

$$\mathbf{E}=-\frac{\partial\mathbf{A}}{\partial t}=-\frac{\mu_0 k}{2}\left(ct-|x|\right)\hat{\mathbf{z}}\ , \tag{6.20}$$

$$\mathbf{B}=\nabla\times\mathbf{A}=-\frac{\mu_0 k}{4c}\frac{\partial}{\partial x}\left(ct-|x|\right)^2\hat{\mathbf{y}}=\pm\frac{\mu_0 k}{2c}\left(ct-|x|\right)\hat{\mathbf{y}}\ , \tag{6.21}$$

where sign plus is for $x > 0$ and minus, for $x < 0$. For $|x| > ct$, $\mathbf{E} = \mathbf{B} = 0$. The electric and magnetic field are shown in Figure 6.1.

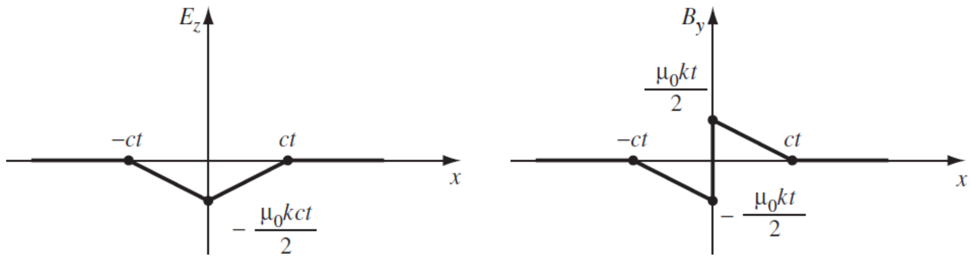

Fig. 6.1

Calculating the derivatives, we find:

$$\nabla\cdot\mathbf{E}=0\,,\quad \nabla\cdot\mathbf{B}=0\,,\quad \nabla\times\mathbf{E}=\mp\frac{\mu_0 k}{2}\hat{\mathbf{y}}\,,\quad \nabla\times\mathbf{B}=-\frac{\mu_0 k}{2c}\hat{\mathbf{z}}\,,\quad \frac{\partial\mathbf{E}}{\partial t}=-\frac{\mu_0 kc}{2}\hat{\mathbf{z}}\,,\quad \frac{\partial\mathbf{B}}{\partial t}=\pm\frac{\mu_0 k}{2}\hat{\mathbf{y}}\,. \tag{6.22}$$

As one can easily check, Maxwell's equations are all satisfied, with ρ and **J** both zero. Notice, however, that **B** has a discontinuity at $x = 0$, and this signals the presence of a surface current **K** in the *yz*-plane. According to the boundary condition (1.81) we have

$$\mathbf{B}_1^{\parallel}-\mathbf{B}_2^{\parallel}=\mu_0\mathbf{K}\times\mathbf{n}\,, \tag{6.23}$$

where **n** is pointing from medium 2 to medium 1. In our case we obtain:

$$kt\hat{\mathbf{y}}=\mathbf{K}\times\hat{\mathbf{x}}\,. \tag{6.24}$$

Evidently, we have here a uniform surface current $\mathbf{K} = kt\hat{\mathbf{z}}$ flowing in the z direction over the plane $x = 0$, which starts up at $t = 0$, and increases in proportion to t. Notice that the news travels out (in both directions) at the speed of light: for points $|x| > ct$ the message has not yet arrived, so the fields are zero.

6.2 Gauge transformations

The transformation (6.13) and (6.14) is called a *gauge transformation*, and the invariance of the fields under such transformations is called gauge invariance. The relation (6.15) between **A** and Φ is called the *Lorenz condition*. To see that potentials can always be found to satisfy the Lorenz condition, suppose that the potentials **A** and Φ that satisfy Eqs. (6.9) and (6.12) do not satisfy Eq. (6.15). Then let us make a gauge transformation to potentials $\mathbf{A}'$ and Φ' and demand that $\mathbf{A}'$ and Φ' satisfy the Lorenz condition:

$$\nabla\cdot\mathbf{A}'+\frac{1}{c^2}\frac{\partial\Phi'}{\partial t}=0=\nabla\cdot\mathbf{A}+\nabla^2\Lambda+\frac{1}{c^2}\frac{\partial\Phi}{\partial t}-\frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2}\,. \tag{6.25}$$

Thus, provided a *gauge function* Λ can be found to satisfy

$$\nabla^2\Lambda-\frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2}=-\left(\nabla\cdot\mathbf{A}+\frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right), \tag{6.26}$$

the new potentials $\mathbf{A}'$ and Φ' will satisfy the Lorenz condition and wave equations (6.16) and (6.17).

Even for potentials that satisfy the Lorenz condition (6.15) there is arbitrariness. Evidently the *restricted gauge transformation*,

$$\mathbf{A}\to\mathbf{A}+\nabla\Lambda\,, \tag{6.27}$$

$$\Phi \to \Phi - \frac{\partial \Lambda}{\partial t}, \tag{6.28}$$

where

$$\nabla^2 \Lambda - \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2} = 0, \tag{6.29}$$

preserves the Lorenz condition, provided **A** and Φ satisfy it initially. All potentials in this restricted class are said to belong to the *Lorenz gauge.* The Lorenz gauge is commonly used, first because it leads to the wave equations (6.9) and (6.12), which treat **A** and Φ on equivalent footings.

Another useful gauge for the potentials is the so-called *Coulomb gauge.* This is the gauge in which

$$\nabla \cdot \mathbf{A} = 0 . \tag{6.30}$$

From (6.9) we see that the scalar potential satisfies Poisson's equation,

$$\nabla^2 \Phi = -\frac{\rho}{\varepsilon_0}, \tag{6.31}$$

with the solution in unrestricted space

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} d^3 r' . \tag{6.32}$$

The scalar potential is just the *instantaneous* Coulomb potential due to the charge density $\rho(\mathbf{r}, t)$. This is the origin of the name "*Coulomb gauge*". There is a peculiar thing about the scalar potential (6.32) in the Coulomb gauge: it is determined by the distribution of charge *right* now. That sounds particularly odd in the light of special relativity, which allows no message to travel faster than the speed of light. The point is that Φ by itself is not a physically measurable quantity – all we can measure is **E**, and that involves **A** as well. Somehow it is built into the vector potential, in the Coulomb gauge, that whereas Φ instantaneously reflects all changes in ρ, the combination $-\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}$ does not; **E** will change only after sufficient time has elapsed for the "news" to arrive.

The *advantage* of the Coulomb gauge is that the *scalar* potential is particularly simple to calculate; the *disadvantage* is that **A** is particularly *difficult* to calculate. The differential equation for **A** (6.12) in the Coulomb gauge reads

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} + \frac{1}{c^2}\nabla\left(\frac{\partial \Phi}{\partial t}\right). \tag{6.33}$$

Example: *Potentials of an infinite sheet of charge*

Assume that the potentials are given by

$$\Phi = 0 , \tag{6.34}$$

$$\mathbf{A}(z, t) = -\frac{\sigma t}{2\varepsilon_0}\hat{\mathbf{z}}, \tag{6.35}$$

where σ is a constant. We need to find fields, and charge and current distributions, which correspond to these potentials.

The electric and magnetic fields are

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} = \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{z}}\,, \tag{6.36}$$

$$\mathbf{B} = \nabla \times \mathbf{A} = \nabla \times \left(-\frac{\sigma}{2\varepsilon_0}\hat{\mathbf{z}} \right) = 0\,. \tag{6.37}$$

We see that these are simply the fields due to an infinite sheet carrying a surface charge *σ*. However, the potentials in form of Eqs. (6.34) and (6.35) are not typical. We can transform these potentials to the conventional form using the gauge function $\Lambda = \frac{\sigma z t}{2\varepsilon_0}$. Using Eqs. (6.27) and (6.28), we find

$$\mathbf{A}' = \mathbf{A} + \nabla\Lambda = -\frac{\sigma t}{2\varepsilon_0}\hat{\mathbf{z}} + \nabla\left(\frac{\sigma z t}{2\varepsilon_0}\right) = 0\,, \tag{6.38}$$

$$\Phi' = \Phi - \frac{\partial \Lambda}{\partial t} = -\frac{\sigma z}{2\varepsilon_0}\,. \tag{6.39}$$

These are the "standard" potentials for an infinite sheet with surface charge density *σ*.

6.3 Retarded potentials

In the Lorenz gauge **A** and Φ satisfy the *inhomogeneous wave equation*, with a "source" term (in place of zero) on the right:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -\frac{\rho}{\varepsilon_0}\,, \tag{6.40}$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J}\,. \tag{6.41}$$

From now on we will use the Lorenz gauge exclusively, and the whole of electrodynamics reduces to the problem of *solving the inhomogeneous wave equations for specified sources.*

In the *static* case, Eqs. (6.40) and (6.41) reduce to Poisson's equation

$$\nabla^2\Phi = -\frac{\rho}{\varepsilon_0}\,, \tag{6.42}$$

$$\nabla^2\mathbf{A} = -\mu_0\mathbf{J}\,, \tag{6.43}$$

with familiar solutions in the unrestricted space

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r'\,, \tag{6.44}$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r'\,. \tag{6.45}$$

In the *dynamic* case, electromagnetic "news" travel at the speed of light. Therefore, it is not the status of the source *right now* that matters, but rather its condition at some earlier time t_r (called the *retarded time***)** when the "message" left. Since this message must travel a distance $|\mathbf{r}-\mathbf{r}'|$, the delay is $|\mathbf{r}-\mathbf{r}'|/c$:

$$t_r \equiv t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\,. \tag{6.46}$$

The natural generalization of Eqs. (6.44) and (6.45) to the dynamic case is therefore

$$\Phi(\mathbf{r},t)=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|}d^3r'\,, \tag{6.47}$$

$$\mathbf{A}(\mathbf{r},t)=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|}d^3r'\,. \tag{6.48}$$

Here $\rho(\mathbf{r}',t_r)$ and $\mathbf{J}(\mathbf{r}',t_r)$ is the charge and current densities that prevailed at point $\mathbf{r}'$ at the retarded time t_r. Because the integrands are evaluated at the retarded time, these are called *retarded potentials*. Note that the retarded potentials are reduced properly to Eqs. (6.44) and (6.45) in the static case, for which ρ and $\mathbf{J}$ are independent of time.

That all sounds reasonable – and surprisingly simple. But so far, we have not proved that all this is correct. To prove this, we must show that the potentials in the form (6.47) and (6.48) satisfy the inhomogeneous wave equations (6.40) and (6.41), and meet the Lorenz condition (6.15). In calculating the Laplacian, we have to take into account that the integrands in Eqs. (6.47) and (6.48) depend on $\mathbf{r}$ in two places: explicitly, in the denominator $|\mathbf{r}-\mathbf{r}'|$, and implicitly, though $t_r=t-|\mathbf{r}-\mathbf{r}'|/c$. Thus,

$$\nabla\Phi(\mathbf{r},t)=\frac{1}{4\pi\varepsilon_0}\int\left[\frac{1}{|\mathbf{r}-\mathbf{r}'|}\nabla\rho(\mathbf{r}',t_r)+\rho(\mathbf{r}',t_r)\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right]d^3r'\,, \tag{6.49}$$

and

$$\nabla\rho(\mathbf{r}',t_r)=\sum_i\hat{\mathbf{x}}_i\frac{\partial\rho}{\partial x_i}=\sum_i\hat{\mathbf{x}}_i\frac{\partial\rho}{\partial t_r}\frac{\partial t_r}{\partial x_i}=\frac{\partial\rho}{\partial t_r}\nabla t_r=-\frac{1}{c}\frac{\partial\rho}{\partial t_r}\nabla|\mathbf{r}-\mathbf{r}'|=-\frac{1}{c}\frac{\partial\rho}{\partial t_r}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\,, \tag{6.50}$$

where we took into account that $\nabla r=\hat{\mathbf{r}}$. Next, using $\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}=-\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}$, we find

$$\nabla\Phi(\mathbf{r},t)=\frac{1}{4\pi\varepsilon_0}\int\left[-\frac{1}{c}\frac{\partial\rho(\mathbf{r}',t_r)}{\partial t_r}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^2}-\rho(\mathbf{r}',t_r)\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\right]d^3r'\,, \tag{6.51}$$

Taking the divergence, we obtain:

$$\nabla^2\Phi(\mathbf{r},t)=\frac{1}{4\pi\varepsilon_0}\int\left\{-\frac{1}{c}\left[\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^2}\cdot\nabla\left(\frac{\partial\rho}{\partial t_r}\right)+\frac{\partial\rho}{\partial t_r}\nabla\cdot\left(\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^2}\right)\right]-\left[\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\cdot\nabla\rho+\rho\nabla\cdot\left(\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\right)\right]\right\}d^3r'\,. \tag{6.52}$$

Similar to Eq. (6.50), we have

$$\nabla\left(\frac{\partial\rho}{\partial t_r}\right)=\frac{\partial^2\rho}{\partial t_r{}^2}\nabla t_r=-\frac{1}{c}\frac{\partial^2\rho}{\partial t_r{}^2}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\,. \tag{6.53}$$

and

$$\nabla\cdot\left(\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^2}\right)=\frac{1}{|\mathbf{r}-\mathbf{r}'|^2}\nabla\cdot(\mathbf{r}-\mathbf{r}')+(\mathbf{r}-\mathbf{r}')\cdot\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|^2}=\frac{3}{|\mathbf{r}-\mathbf{r}'|^2}+(\mathbf{r}-\mathbf{r}')\cdot(-2)\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^4}=\frac{1}{|\mathbf{r}-\mathbf{r}'|^2}\,. \tag{6.54}$$

In addition, we know that

$$\nabla\cdot\left(\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\right)=4\pi\delta^3(\mathbf{r}-\mathbf{r}')\,. \tag{6.55}$$

Due to Eqs. (6.50) and (6.54), the second and third terms in Eq. (6.52) are canceled out. Thus,

$$\nabla^2\Phi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int\left\{\frac{1}{c^2}\frac{\partial^2\rho(\mathbf{r},t_r)}{\partial t_r{}^2}\frac{1}{|\mathbf{r}-\mathbf{r}'|} - 4\pi\rho(\mathbf{r}',t_r)\delta^3(\mathbf{r}-\mathbf{r}')\right\}d^3r' = \frac{1}{c^2}\frac{\partial^2\Phi(\mathbf{r},t)}{\partial t^2} - \frac{1}{\varepsilon_0}\rho(\mathbf{r},t)\,, \quad (6.56)$$

where we took into account that $\frac{\partial^2}{\partial t_r{}^2} = \frac{\partial^2}{\partial t^2}$ and that t_r in the second term should be replaced by t due to the delta function. Eq. (6.56) confirms that that the retarded potential (6.47) satisfies the inhomogeneous wave equation (6.40). Similar derivation can be performed for the vector potential.

Now let us demonstrate that the retarded potentials satisfy the Lorenz gauge condition $\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t} = 0$. We need to calculate divergence $\mathbf{A}$ given by Eq. (6.45) and therefore we have to find $\nabla\cdot\left(\frac{\mathbf{J}(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|}\right)$. Let us rewrite it in the following way:

$$\begin{aligned}\nabla\cdot\left(\frac{\mathbf{J}}{|\mathbf{r}-\mathbf{r}'|}\right) &= \frac{1}{|\mathbf{r}-\mathbf{r}'|}(\nabla\cdot\mathbf{J}) + \mathbf{J}\cdot\left(\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = \frac{1}{|\mathbf{r}-\mathbf{r}'|}(\nabla\cdot\mathbf{J}) - \mathbf{J}\cdot\left(\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = \\ &= \frac{1}{|\mathbf{r}-\mathbf{r}'|}(\nabla\cdot\mathbf{J}) + \frac{1}{|\mathbf{r}-\mathbf{r}'|}(\nabla'\cdot\mathbf{J}) - \nabla'\cdot\left(\frac{\mathbf{J}}{|\mathbf{r}-\mathbf{r}'|}\right).\end{aligned} \quad (6.57)$$

Here ∇' denotes differentiation with respect to $\mathbf{r}'$, and we took into account that $\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|}$.

Then we find

$$\nabla\cdot\mathbf{J}(\mathbf{r}',t_r) = \sum_i\frac{\partial J_i(\mathbf{r}',t_r)}{\partial x_i} = \sum_i\frac{\partial J_i}{\partial t_r}\frac{\partial t_r}{\partial x_i} = \frac{\partial\mathbf{J}}{\partial t_r}\cdot\nabla t_r = -\frac{1}{c}\frac{\partial\mathbf{J}}{\partial t_r}\cdot\nabla|\mathbf{r}-\mathbf{r}'|\,. \quad (6.58)$$

Similarly

$$\nabla'\cdot\mathbf{J}(\mathbf{r}',t_r) = \sum_i\left(\frac{\partial J_i}{\partial x_i'} + \frac{\partial J_i}{\partial t_r}\frac{\partial t_r}{\partial x_i'}\right) = \tilde{\nabla}'\cdot\mathbf{J} + \frac{\partial\mathbf{J}}{\partial t_r}\cdot\nabla' t_r = -\frac{\partial\rho}{\partial t} - \frac{1}{c}\frac{\partial\mathbf{J}}{\partial t_r}\cdot\nabla'|\mathbf{r}-\mathbf{r}'|\,. \quad (6.59)$$

In Eq. (6.59), the first term arises when we differentiate with respect to the explicit $\mathbf{r}'$ (at a fixed t_r, as denoted by $\tilde{\nabla}'$) and use the continuity equation. Thus,

$$\begin{aligned}\nabla\cdot\left(\frac{\mathbf{J}}{|\mathbf{r}-\mathbf{r}'|}\right) &= -\frac{1}{|\mathbf{r}-\mathbf{r}'|}\frac{1}{c}\frac{\partial\mathbf{J}}{\partial t_r}\cdot\nabla|\mathbf{r}-\mathbf{r}'| + \frac{1}{|\mathbf{r}-\mathbf{r}'|}\left(-\frac{\partial\rho}{\partial t} - \frac{1}{c}\frac{\partial\mathbf{J}}{\partial t_r}\cdot\nabla'|\mathbf{r}-\mathbf{r}'|\right) - \nabla'\cdot\left(\frac{\mathbf{J}}{|\mathbf{r}-\mathbf{r}'|}\right) = \\ &= -\frac{1}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial\rho}{\partial t} - \nabla'\cdot\left(\frac{\mathbf{J}}{|\mathbf{r}-\mathbf{r}'|}\right),\end{aligned} \quad (6.60)$$

where we took into account that $\nabla|\mathbf{r}-\mathbf{r}'| = -\nabla'|\mathbf{r}-\mathbf{r}'|$. Finally, we find

$$\begin{aligned}\nabla\cdot\mathbf{A}(\mathbf{r},t) &= \frac{\mu_0}{4\pi}\int\nabla\cdot\left[\frac{\mathbf{J}(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|}\right]d^3r' = \frac{\mu_0}{4\pi}\left\{-\int\frac{1}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial\rho(\mathbf{r}',t_r)}{\partial t}d^3r' - \int\nabla'\cdot\left[\frac{\mathbf{J}(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|}\right]d^3r'\right\} = \\ &\frac{\mu_0}{4\pi}\left\{-\frac{\partial}{\partial t}\int\frac{\rho(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|}d^3r' - \oint\frac{\mathbf{J}(\mathbf{r}',t_r)\cdot\mathbf{n}}{|\mathbf{r}-\mathbf{r}'|}da\right\} = -\mu_0\varepsilon_0\frac{\partial}{\partial t}\left\{\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|}d^3r'\right\} = -\frac{1}{c^2}\frac{\partial\Phi(\mathbf{r},t)}{\partial t}.\end{aligned} \quad (6.61)$$

Here we assumed that $\mathbf{J}$ = 0 at infinity and therefore the surface integral vanished. Thus, we have proved that the Lorenz gauge condition is satisfied for the retarded potentials.

Incidentally, this proof applies equally well to the *advanced potentials*,

$$\Phi_a(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}',t_a)}{|\mathbf{r}-\mathbf{r}'|}d^3r', \tag{6.62}$$

$$\mathbf{A}_a(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}',t_a)}{|\mathbf{r}-\mathbf{r}'|}d^3r'. \tag{6.63}$$

in which the charge and the current densities are evaluated at the *advanced time*

$$t_a = t + \frac{|\mathbf{r}-\mathbf{r}'|}{c}. \tag{6.64}$$

A few signs are changed, but the final result is unaffected. Although the advanced potentials are entirely consistent with Maxwell's equations, they violate the most sacred tenet in all of physics: the principle of *causality*. They suggest that the potentials *now* depend on what the charge and the current distribution *will* be at some time in the future – the effect, in other words, precedes the cause. Although the advanced potentials are of some theoretical interest, they have no direct physical significance.

Example: An infinite straight wire carries the current

$$I(t) = \begin{cases} 0, & t<0 \\ I_0, & t>0 \end{cases} \tag{6.65}$$

That is, a constant current I_0 is turned on abruptly at t = 0. Find the resulting electric and magnetic fields.

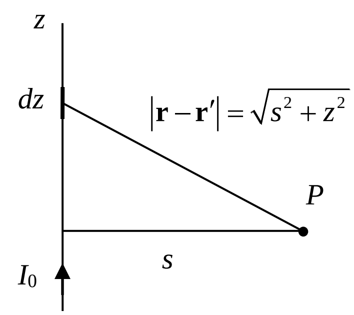

Fig. 6.2

Solution: The wire is electrically neutral, so that the scalar potential is zero. Let the wire lie along the z axis as is shown in figure. The retarded potential at point P is

$$\mathbf{A}(s,t) = \frac{\mu_0}{4\pi}\hat{\mathbf{z}}\int_{-\infty}^{\infty}\frac{I(t_r)}{|\mathbf{r}-\mathbf{r}'|}dz'. \tag{6.66}$$

For $t < s/c$, the information about current flowing in the wire has not yet reached P, and the potential is zero. For $t > s/c$, only the segment $|z| \le \sqrt{(ct)^2 - s^2}$ contributes (outside this range t_r is negative, so $I(t_r) = 0$). Thus

$$\mathbf{A}(s,t) = \left(\frac{\mu_0 I_0}{4\pi}\hat{\mathbf{z}}\right)2\int_0^{\sqrt{(ct)^2-s^2}}\frac{dz}{\sqrt{s^2+z^2}} = \frac{\mu_0 I_0}{2\pi}\hat{\mathbf{z}}\ln\left(\sqrt{s^2+z^2}+z\right)\Bigg|_0^{\sqrt{(ct)^2-s^2}} = \frac{\mu_0 I_0}{2\pi}\ln\left(\frac{ct+\sqrt{(ct)^2-s^2}}{s}\right)\hat{\mathbf{z}}. \tag{6.67}$$

The electric field is

$$\mathbf{E}(s,t) = -\frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 I_0}{2\pi}\frac{s}{ct+\sqrt{(ct)^2-s^2}}\frac{1}{s}\left(c+\frac{2c^2 t}{2\sqrt{(ct)^2-s^2}}\right)\hat{\mathbf{z}} = -\frac{\mu_0 I_0 c}{2\pi\sqrt{(ct)^2-s^2}}\hat{\mathbf{z}}. \tag{6.68}$$

The magnetic field is

$$\mathbf{B}(s,t)=\nabla\times\mathbf{A}=-\frac{\partial A_z}{\partial s}\hat{\boldsymbol{\phi}}=$$
$$=-\frac{\mu_0 I_0}{2\pi}\frac{s}{ct+\sqrt{(ct)^2-s^2}}\left(-\frac{ct+\sqrt{(ct)^2-s^2}}{s^2}-\frac{1}{\sqrt{(ct)^2-s^2}}\right)\hat{\boldsymbol{\phi}}=\frac{\mu_0 I_0}{2\pi s}\frac{ct}{\sqrt{(ct)^2-s^2}}\hat{\boldsymbol{\phi}}. \tag{6.69}$$

Note that the fields are non-zero only when $ct > s$. Notice that as $t\to\infty$ we recover the known result in static case:

$$\mathbf{E}=0\,, \tag{6.70}$$

$$\mathbf{B}=\frac{\mu_0 I_0}{2\pi s}\hat{\boldsymbol{\phi}}\,. \tag{6.71}$$

6.4 Retarded fields

Given the retarded potentials

$$\Phi(\mathbf{r},t)=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|}d^3r'\,, \tag{6.72}$$

$$\mathbf{A}(\mathbf{r},t)=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|}d^3r'\,, \tag{6.73}$$

it is straightforward matter to calculate the fields

$$\mathbf{B}=\nabla\times\mathbf{A}\,, \tag{6.74}$$

$$\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}\,. \tag{6.75}$$

For the electric field **E**, we have already calculated the gradient of Φ (Eq. (6.51)). The time derivative of **A** is given by

$$\frac{\partial\mathbf{A}}{\partial t}=\frac{\mu_0}{4\pi}\int\frac{\dot{\mathbf{J}}}{|\mathbf{r}-\mathbf{r}'|}d^3r'\,, \tag{6.76}$$

where the overdot indicates differentiation with respect to *t*. Putting together we find

$$\mathbf{E}(\mathbf{r},t)=\frac{1}{4\pi\varepsilon_0}\int\left[\rho(\mathbf{r}',t_r)\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}+\frac{1}{c}\dot{\rho}(\mathbf{r}',t_r)\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^2}-\frac{1}{c^2}\frac{\dot{\mathbf{J}}(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|}\right]d^3r'\,, \tag{6.77}$$

This is the time-dependent generalization of Coulomb's law, to which it reduces in the static case (where the second and third terms drop out and the first term loses its dependence on t_r).

As for the magnetic field **B**, the curl of **A** contains two terms:

$$\nabla\times\mathbf{A}=\frac{\mu_0}{4\pi}\int\left[\frac{1}{|\mathbf{r}-\mathbf{r}'|}(\nabla\times\mathbf{J})-\mathbf{J}\times\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right]d^3r'. \tag{6.78}$$

We can represent the *i*-component of the curl of **J** as follows:

$$\left(\nabla\times\mathbf{J}\right)_i=\sum_{jk}\varepsilon_{ijk}\frac{\partial J_k}{\partial x_j}=\sum_{jk}\varepsilon_{ijk}\frac{\partial J_k}{\partial t_r}\frac{\partial t_r}{\partial x_j}=-\frac{1}{c}\sum_{jk}\varepsilon_{ijk}\dot{J}_k\frac{\partial\left|\mathbf{r}-\mathbf{r}'\right|}{\partial x_j}=\frac{1}{c}\left(\dot{\mathbf{J}}\times\nabla\left|\mathbf{r}-\mathbf{r}'\right|\right)_i. \tag{6.79}$$

Here we used the *Levi-Civita* symbol ε_{ijk} which is defined as follows:

$$\varepsilon_{ijk}=\begin{cases}+1, & \text{if } ikj=123,\ 231,\text{ or }312\\ -1, & \text{if } ikj=132, 213,\text{ or }321\\ 0, & \text{otherwise.}\end{cases} \tag{6.80}$$

in terms of which the cross product can be written as $\left(\mathbf{a}\times\mathbf{b}\right)_i=\sum_{jk}\varepsilon_{ijk}a_jb_k$.

Since $\nabla\left|\mathbf{r}-\mathbf{r}'\right|=\frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|}$, we find

$$\nabla\times\mathbf{J}=\frac{1}{c}\dot{\mathbf{J}}\times\frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|}. \tag{6.81}$$

Taking into account $\nabla\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|}=-\frac{\left(\mathbf{r}-\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|^3}$, we obtain

$$\mathbf{B}(\mathbf{r},t)=\frac{\mu_0}{4\pi}\int\left[\frac{\mathbf{J}(\mathbf{r}',t_r)}{\left|\mathbf{r}-\mathbf{r}'\right|^3}+\frac{\dot{\mathbf{J}}(\mathbf{r}',t_r)}{c\left|\mathbf{r}-\mathbf{r}'\right|^2}\right]\times\left(\mathbf{r}-\mathbf{r}'\right)d^3r'. \tag{6.82}$$

This is time dependent generalization of Bio-Savart law, to which it reduces in the static case. Equations (6.77) and (6.82) are known as *Jefimenko's equations*. In practice Jefimenko's equations are of limited utility, since it is typically easier to calculate the retarded potentials and differentiate them, rather than going directly to the fields. Nevertheless, they provide a satisfying sense of closure to the theory. They also help to clarify the following observation: To get to the retarded *potentials*, all you do is replace t by t_r in the electrostatic and magnetostatic formulas, but in the case of *the fields* not only is time replaced by retarded time, but completely new terms (involving derivatives of ρ and **J**) appear.

Example: Calculate the electric and magnetic fields in the example above using Jefimenko's equations.

We start from Eq. (6.77). Since $\rho=0$, the first two terms in the integral are zero and hence

$$\mathbf{E}(\mathbf{r},t)=-\frac{1}{4\pi\varepsilon_0}\int\frac{1}{c^2}\frac{\dot{\mathbf{J}}(\mathbf{r}',t_r)}{\left|\mathbf{r}-\mathbf{r}'\right|}d^3r'. \tag{6.83}$$

Now the current has form $I(t)=I_0\Theta(t)$ where $\Theta(t)$ is the unit step function $\Theta(t)=\begin{cases}0, & t<0\\ 1, & t>0\end{cases}$, so that its time derivative is the delta function so that $\dot{I}(t)=I_0\delta(t)$. Therefore, we obtain for the field

$$\mathbf{E}(\mathbf{r},t)=-\frac{I_0\hat{\mathbf{z}}}{4\pi\varepsilon_0c^2}\int_{-\infty}^{\infty}\frac{\delta(t_r)}{\left|\mathbf{r}-\mathbf{r}'\right|}dz=-\frac{I_0\hat{\mathbf{z}}}{4\pi\varepsilon_0c}\int_{-\infty}^{\infty}\frac{\delta\left(\xi-ct\right)}{\xi}dz, \tag{6.84}$$

where we defined $\xi\equiv\left|\mathbf{r}-\mathbf{r}'\right|$ and used $\delta(t_r)=\delta\left(t-\frac{\xi}{c}\right)=c\delta\left(\xi-ct\right)$. Since $\left|\mathbf{r}-\mathbf{r}'\right|=\sqrt{s^2+z^2}$ (see Fig. 5.2), we have $\xi d\xi=zdz$ and $z=\sqrt{\xi^2-s^2}$. Changing the variable of integration in Eq. (6.84),

$$\mathbf{E}(\mathbf{r},t) = -\frac{I_0\hat{\mathbf{z}}}{4\pi\varepsilon_0 c} 2\int_s^\infty \frac{\delta(\xi - ct)}{\xi} \frac{\xi d\xi}{\sqrt{\xi^2 - s^2}} = \begin{cases} 0, & ct < s, \\ -\dfrac{I_0\hat{\mathbf{z}}}{2\pi\varepsilon_0 c\sqrt{(ct)^2 - s^2}}, & ct > s. \end{cases} \tag{6.85}$$

Here a factor of two appeared due to the integration over positive and negative *z*.

The magnetic field is given by Eq. (6.82), which in our case takes form

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_0 I_0}{4\pi} \int_{-\infty}^{\infty} \left[\frac{\Theta(t_r)}{|\mathbf{r} - \mathbf{r}'|^3} + \frac{\delta(t_r)}{c|\mathbf{r} - \mathbf{r}'|^2} \right] \hat{\mathbf{z}} \times (\mathbf{r} - \mathbf{r}') dz \,. \tag{6.86}$$

Using $\hat{\mathbf{z}} \times (\mathbf{r} - \mathbf{r}') = s\hat{\boldsymbol{\phi}}$ and the definitions above, we obtain

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_0 I_0 s\hat{\boldsymbol{\phi}}}{2\pi} \int_s^\infty \left[\frac{\Theta(ct - \xi)}{\xi^3} + \frac{\delta(\xi - ct)}{\xi^2} \right] \frac{\xi d\xi}{\sqrt{\xi^2 - s^2}} \,, \tag{6.87}$$

where we used $\Theta(t_r) = \Theta(ct - \xi)$.

Now using $\Theta(ct - \xi) = \begin{cases} 0, & \xi > ct \\ 1, & \xi < ct \end{cases}$, we obtain for the first integral:

$$\int_s^\infty \frac{\Theta(ct - \xi)}{\xi^2} \frac{d\xi}{\sqrt{\xi^2 - s^2}} = \int_s^{ct} \frac{1}{\xi^2} \frac{d\xi}{\sqrt{\xi^2 - s^2}} = \left. \frac{\sqrt{\xi^2 - s^2}}{s^2 \xi} \right|_s^{ct} = \frac{\sqrt{(ct)^2 - s^2}}{s^2 ct} \,. \tag{6.88}$$

For the second integral we obtain:

$$\int_s^\infty \frac{\delta(\xi - ct)}{\xi^2} \frac{\xi d\xi}{\sqrt{\xi^2 - s^2}} = \frac{1}{ct\sqrt{(ct)^2 - s^2}} \,. \tag{6.89}$$

Both integrals are non-zero only for $ct > s$. Finally, we have for the magnetic field

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_0 I_0 s\hat{\boldsymbol{\phi}}}{2\pi} \left[\frac{\sqrt{(ct)^2 - s^2}}{s^2 ct} + \frac{1}{ct\sqrt{(ct)^2 - s^2}} \right] = \begin{cases} 0, & ct < s, \\ \dfrac{\mu_0 I_0}{2\pi s} \dfrac{ct}{\sqrt{(ct)^2 - s^2}} \hat{\boldsymbol{\phi}}, & ct > s. \end{cases} \tag{6.90}$$

Equations (6.85) and (6.90) for the fields are identical to those obtained is section 5.3 using the potential formulation of Maxwell's equations.

6.5 Potentials of a moving point charge

Next, we calculate the (retarded) potentials, $\Phi(\mathbf{r},t)$ and $\mathbf{A}(\mathbf{r},t)$, of a point charge *q* that is moving on a specified trajectory $\mathbf{w}(t)$. A naïve reading of the formula (6.47)

$$\Phi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{r}',t_r)}{|\mathbf{r} - \mathbf{r}'|} d^3 r' \,, \tag{6.91}$$

might suggest to you that the potential is simply $\Phi(\mathbf{r},t) = \dfrac{q}{4\pi\varepsilon_0 |\mathbf{r} - \mathbf{w}(t_r)|}$, i.e. the same as in the static case, but with the understanding that $\mathbf{r}$ is the distance to the retarded position of the charge $\mathbf{w}(t_r)$. But this is wrong, for a very subtle reason. While it is true that for a point source, the denominator r comes outside the integral, resulting in a factor $1/|\mathbf{r}-\mathbf{w}(t_r)|$, but the remaining integral $\int \rho(\mathbf{r}',t_r)d^3r'$ appears to be not equal to the charge q and dependent, through t_r, on the location of the point $\mathbf{r}$.

To demonstrate this explicitly, we write the charge density (entering Eq. (6.91)) of a point charge in terms of a delta function in space, so if the charge's trajectory is given by $\mathbf{w}(t')$ then

$$\rho(\mathbf{r}',t') = q\delta^3\left(\mathbf{r}'-\mathbf{w}(t')\right). \tag{6.92}$$

To work out Φ, we need the charge density at the retarded time t_r, which we can write as the integral over time of the charge density multiplied by another delta function:

$$\rho(\mathbf{r}',t_r) = q\int \delta^3\left(\mathbf{r}'-\mathbf{w}(t')\right)\delta\left(t'-t_r\right)dt'. \tag{6.93}$$

We need to keep straight the different times we are using here. The time t is the observation time, t' is the integration variable, and t_r is the retarded time (Fig. 6.3), i.e. is the time at which the signal that we are receiving at time t left the moving charge, that is

$$t_r = t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}. \tag{6.94}$$

The potential can now be written as an integral over both time and space:

$$\Phi(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0}\int\int \frac{\delta^3\left(\mathbf{r}'-\mathbf{w}(t')\right)}{|\mathbf{r}-\mathbf{r}'|}\delta\left[t'-\left(t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\right]d^3r'dt'. \tag{6.95}$$

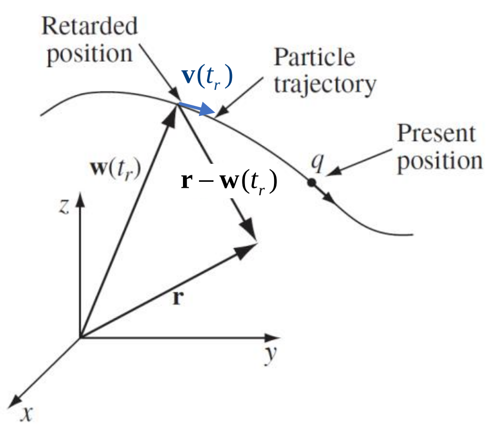

Fig. 6.3

We can do the spatial integration which sets $\mathbf{r}'=\mathbf{w}(t')$, resulting in

$$\Phi(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0}\int \frac{dt'}{|\mathbf{r}-\mathbf{w}(t')|}\delta\left[t'-\left(t-\frac{|\mathbf{r}-\mathbf{w}(t')|}{c}\right)\right]. \tag{6.96}$$

The trick now is to transform the argument of the delta function so we can perform the integration. To do this, we need to work out $\delta\left(f(x)\right)$ for some function $f(x)$. We consider

$$u = f(x), \tag{6.97}$$

$$du = f'(x)dx, \tag{6.98}$$

so that

$$\int \delta\left(f(x)\right)dx = \int \frac{\delta(u)}{|f'(x)|}du = \frac{1}{\left|f'\left(x(0)\right)\right|}, \tag{6.99}$$

where the absolute value of the derivative $f'(x)$ results from the fact that the limits of integration near the root of $u = 0$ need to be changed to the opposite if $f'(x) < 0$. Therefore, we need to solve Eq. (6.97) for x as a function of u and then find x ($u = 0$).

For our problem, we have

$$f(t') = t' - \left(t - \frac{|\mathbf{r} - \mathbf{w}(t')|}{c}\right), \tag{6.100}$$

$$\frac{df}{dt'} = 1 + \frac{1}{c}\frac{d}{dt'}|\mathbf{r} - \mathbf{w}(t')|. \tag{6.101}$$

The latter derivative can be calculated as follows. Denoting $\mathbf{u}(t') \equiv \mathbf{r} - \mathbf{w}(t')$, we have

$$\frac{d|\mathbf{u}(t')|}{dt'} = \frac{d\sqrt{\mathbf{u}\cdot\mathbf{u}}}{dt'} = \frac{1}{2}\frac{1}{\sqrt{\mathbf{u}\cdot\mathbf{u}}}\frac{d(\mathbf{u}\cdot\mathbf{u})}{dt'} = \frac{1}{2|\mathbf{u}|}2\mathbf{u}\cdot\frac{d\mathbf{u}}{dt'} = \frac{\mathbf{u}}{|\mathbf{u}|}\cdot\frac{d\mathbf{u}}{dt'}. \tag{6.102}$$

Therefore, we obtain from Eqs. (6.101) and (6.102):

$$\frac{df}{dt'} = 1 + \frac{1}{c}\frac{d}{dt'}|\mathbf{r} - \mathbf{w}(t')| = 1 - \frac{1}{c}\frac{(\mathbf{r} - \mathbf{w}(t'))\cdot\mathbf{v}(t')}{|\mathbf{r} - \mathbf{w}(t')|}, \tag{6.103}$$

where $\mathbf{v}(t') \equiv \dfrac{d\mathbf{w}(t')}{dt'}$ is the velocity of the charge (Fig. 6.3).

Returning to Eq. (6.96), we notice that, according to Eq. (6.100), $f(t') = 0$ at $t' = t_r$, where the retarded time is determined by

$$t_r = t - \frac{|\mathbf{r} - \mathbf{w}(t_r)|}{c}, \tag{6.104}$$

so that the integral over the delta function results in

$$\Phi(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0}\frac{1}{|\mathbf{r} - \mathbf{w}(t_r)|}\frac{1}{1 - \dfrac{1}{c}\dfrac{(\mathbf{r} - \mathbf{w}(t_r))\cdot\mathbf{v}(t_r)}{|\mathbf{r} - \mathbf{w}(t_r)|}} = \frac{1}{4\pi\varepsilon_0}\frac{qc}{c|\mathbf{r} - \mathbf{w}(t_r)| - (\mathbf{r} - \mathbf{w}(t_r))\cdot\mathbf{v}(t_r)}. \tag{6.105}$$

The current density for a moving point charge is just $\mathbf{J} = \rho\mathbf{v}$, so the derivation of the vector potential $\mathbf{A}$ from Eq. (6.48) follows exactly the same path, and we obtain

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}',t_r)}{|\mathbf{r} - \mathbf{r}'|}d^3r' = \frac{\mu_0}{4\pi}\frac{qc\mathbf{v}(t_r)}{c|\mathbf{r} - \mathbf{w}(t_r)| - (\mathbf{r} - \mathbf{w}(t_r))\cdot\mathbf{v}(t_r)} = \frac{\mathbf{v}}{c^2}\Phi(\mathbf{r},t). \tag{6.106}$$

Equations (6.105) and (6.106) represent the *Liénard-Wiechert potentials* for a moving point charge.

Example: Potentials of a point charge moving with constant velocity.

For convenience, assume that the particle passes through the origin at time $t = 0$, so that

$$\mathbf{w}(t) = \mathbf{v}t. \tag{6.107}$$

We first obtain the retarded time, using Eq. (6.104):

$$|\mathbf{r}-\mathbf{v}t_r| = c(t-t_r)\,. \tag{6.108}$$

Squaring, we have from Eq. (6.108):

$$r^2 - 2\mathbf{r}\cdot\mathbf{v}t_r + v^2 t_r^2 = c^2\left(t^2 - 2tt_r + t_r^2\right). \tag{6.109}$$

Solving for t_r by the quadratic formula, we find

$$t_r = \frac{c^2 t - \mathbf{r}\cdot\mathbf{v} \pm \sqrt{\left(c^2 t - \mathbf{r}\cdot\mathbf{v}\right)^2 + \left(c^2 - v^2\right)\left(r^2 - c^2 t^2\right)}}{c^2 - v^2}\,. \tag{6.110}$$

To fix the sign, consider the limit $v = 0$:

$$t_r = t \pm \frac{r}{c}\,. \tag{6.111}$$

In this case, the charge is at rest at the origin, and the retarded time should be $(t - r/c)$; evidently, we want the *minus* sign.

Now for the denominator of Eq. (6.105), we obtain using Eqs. (6.108) and (6.110)

$$\begin{aligned} c|\mathbf{r}-\mathbf{w}(t_r)| - (\mathbf{r}-\mathbf{w}(t_r))\cdot\mathbf{v}(t_r) &= c|\mathbf{r}-\mathbf{v}t_r| + (\mathbf{r}-\mathbf{v}t_r)\cdot\mathbf{v} = c^2(t-t_r) - \mathbf{r}\cdot\mathbf{v} + v^2 t_r = \\ &= c^2 t - \mathbf{r}\cdot\mathbf{v} - \left(c^2 - v^2\right)t_r = \sqrt{\left(c^2 t - \mathbf{r}\cdot\mathbf{v}\right)^2 + \left(c^2 - v^2\right)\left(r^2 - c^2 t^2\right)}\,. \end{aligned} \tag{6.112}$$

Finally, we obtain for the potentials:

$$\Phi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\frac{qc}{\sqrt{\left(c^2 t - \mathbf{r}\cdot\mathbf{v}\right)^2 + \left(c^2 - v^2\right)\left(r^2 - c^2 t^2\right)}}\,, \tag{6.113}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\frac{qc\mathbf{v}}{\sqrt{\left(c^2 t - \mathbf{r}\cdot\mathbf{v}\right)^2 + \left(c^2 - v^2\right)\left(r^2 - c^2 t^2\right)}}\,. \tag{6.114}$$

The scalar (and also vector) potential of a point charge moving with constant velocity (Eq. (6.113)) can be written in a more simple form (HW#8, Problem 2):

$$\Phi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\frac{q}{R\sqrt{1 - \dfrac{v^2}{c^2}\sin^2\theta}}\,, \tag{6.115}$$

where $\mathbf{R} \equiv \mathbf{r} - \mathbf{v}t$ is the vector from the *present* position of the particle to the field point $\mathbf{r}$ and θ is the angle between $\mathbf{R}$ and $\mathbf{v}$ (Fig. 6.4). Note that $\mathbf{R}$ and θ are both functions of time since they vary as the charge moves

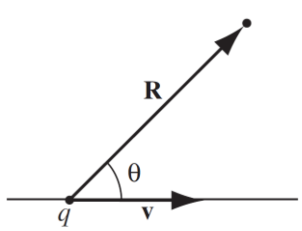

Fig. 6.4

For non-relativistic speeds, $v \ll c$, this expression is reduced to the Coulomb potential in electrostatics:

$$\Phi(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0 R}.$$

6.6 Fields of a moving point charge

We are now in a position to calculate the electric and magnetic fields,

$$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}, \tag{6.116}$$

of a point charge in arbitrary motion, using the Liénard-Wiechert potentials (6.105) and (6.106). We define $\boldsymbol{\xi} \equiv \mathbf{r} - \mathbf{w}(t_r)$ and rewrite these expressions as follows:

$$\Phi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0} \frac{qc}{\xi c - \boldsymbol{\xi}\cdot\mathbf{v}}, \tag{6.117}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mathbf{v}}{c^2}\Phi(\mathbf{r},t). \tag{6.118}$$

Here both $\boldsymbol{\xi}$ and $\mathbf{v} = \dot{\mathbf{w}}$ are to be evaluated at the retarded time t_r, which is defined implicitly by Eq. (6.104), which can be written as

$$\xi = c\left(t - t_r\right). \tag{6.119}$$

First, we calculate $\nabla\Phi$:

$$\nabla\Phi = \frac{qc}{4\pi\varepsilon_0} \frac{(-1)}{\left(\xi c - \boldsymbol{\xi}\cdot\mathbf{v}\right)^2} \nabla\left(\xi c - \boldsymbol{\xi}\cdot\mathbf{v}\right). \tag{6.120}$$

From Eq. (6.119), we have

$$\nabla\xi = -c\nabla t_r. \tag{6.121}$$

For the second term, the 4-th product rule gives

$$\nabla\left(\boldsymbol{\xi}\cdot\mathbf{v}\right) = \left(\boldsymbol{\xi}\cdot\nabla\right)\mathbf{v} + \left(\mathbf{v}\cdot\nabla\right)\boldsymbol{\xi} + \boldsymbol{\xi}\times\left(\nabla\times\mathbf{v}\right) + \mathbf{v}\times\left(\nabla\times\boldsymbol{\xi}\right). \tag{6.122}$$

We evaluate these terms one at a time:

$$\left(\boldsymbol{\xi}\cdot\nabla\right)\mathbf{v} = \sum_i \xi_i \frac{\partial \mathbf{v}(t_r)}{\partial x_i} = \frac{\partial \mathbf{v}}{\partial t_r} \sum_i \xi_i \frac{\partial t_r}{\partial x_i} = \mathbf{a}\left(\boldsymbol{\xi}\cdot\nabla t_r\right), \tag{6.123}$$

where $\mathbf{a} \equiv \dot{\mathbf{v}}(t_r)$ is the acceleration of the particle at the retarded time.

The next term in Eq. (6.122) is

$$\left(\mathbf{v}\cdot\nabla\right)\boldsymbol{\xi} = \left(\mathbf{v}\cdot\nabla\right)\mathbf{r} - \left(\mathbf{v}\cdot\nabla\right)\mathbf{w} = \sum_i v_i \frac{\partial \mathbf{r}}{\partial x_i} - \sum_i v_i \frac{\partial \mathbf{w}(t_r)}{\partial x_i} = \sum_i v_i \hat{\mathbf{x}}_i - \frac{\partial \mathbf{w}}{\partial t_r} \sum_i v_i \frac{\partial t_r}{\partial x_i} = \mathbf{v} - \mathbf{v}\left(\mathbf{v}\cdot\nabla t_r\right). \tag{6.124}$$

Moving on to the third term in Eq. (6.122),

$$\left(\nabla\times\mathbf{v}\right) = \sum_{ijk} \varepsilon_{ijk} \frac{\partial v_k}{\partial x_j}\hat{\mathbf{x}}_i = \sum_{ijk} \varepsilon_{ijk} \frac{\partial v_k}{\partial t_r}\frac{\partial t_r}{\partial x_j}\hat{\mathbf{x}}_i = \sum_{ijk} \varepsilon_{ijk} a_k \frac{\partial t_r}{\partial x_j}\hat{\mathbf{x}}_i = -\mathbf{a}\times\nabla t_r. \tag{6.125}$$

The fourth term gives:

$$\left(\nabla\times\boldsymbol{\xi}\right)=\nabla\times\mathbf{r}-\nabla\times\mathbf{w}=-\sum_{ijk}\varepsilon_{ijk}\frac{\partial w_k}{\partial x_j}\hat{\mathbf{x}}_i=-\sum_{ijk}\varepsilon_{ijk}\frac{\partial w_k}{\partial t_r}\frac{\partial t_r}{\partial x_j}\hat{\mathbf{x}}_i=-\sum_{ijk}\varepsilon_{ijk}v_k\frac{\partial t_r}{\partial x_j}\hat{\mathbf{x}}_i=\mathbf{v}\times\nabla t_r\,. \tag{6.126}$$

Putting all this back into Eq. (6.122), and using the "BAC-CAB" rule to reduce the triple cross products,

$$\nabla\left(\boldsymbol{\xi}\cdot\mathbf{v}\right)=\mathbf{a}\left(\boldsymbol{\xi}\cdot\nabla t_r\right)+\mathbf{v}-\mathbf{v}\left(\mathbf{v}\cdot\nabla t_r\right)-\boldsymbol{\xi}\times\left(\mathbf{a}\times\nabla t_r\right)+\mathbf{v}\times\left(\mathbf{v}\times\nabla t_r\right)=\mathbf{v}+\left(\boldsymbol{\xi}\cdot\mathbf{a}-v^2\right)\nabla t_r\,. \tag{6.127}$$

Collecting Eqs. (6.121) and (6.127), we have

$$\nabla\Phi=\frac{qc}{4\pi\varepsilon_0}\frac{1}{\left(\xi c-\boldsymbol{\xi}\cdot\mathbf{v}\right)^2}\left[\mathbf{v}+\left(c^2-v^2+\boldsymbol{\xi}\cdot\mathbf{a}\right)\nabla t_r\right]. \tag{6.128}$$

To complete the calculation, we need to know ∇t_r. Using Eq. (6.121) and expanding $\nabla\xi$, we find:

$$-c\nabla t_r=\nabla\xi=\nabla\sqrt{\boldsymbol{\xi}\cdot\boldsymbol{\xi}}=\frac{1}{2\sqrt{\boldsymbol{\xi}\cdot\boldsymbol{\xi}}}\nabla\left(\boldsymbol{\xi}\cdot\boldsymbol{\xi}\right)=\frac{1}{\xi}\left[\left(\boldsymbol{\xi}\cdot\nabla\right)\boldsymbol{\xi}+\boldsymbol{\xi}\times\left(\nabla\times\boldsymbol{\xi}\right)\right]. \tag{6.129}$$

But from Eq. (6.126), $\left(\nabla\times\boldsymbol{\xi}\right)=\mathbf{v}\times\nabla t_r$, and similar to Eq. (6.124), we have

$$\left(\boldsymbol{\xi}\cdot\nabla\right)\boldsymbol{\xi}=\left(\boldsymbol{\xi}\cdot\nabla\right)\mathbf{r}-\left(\boldsymbol{\xi}\cdot\nabla\right)\mathbf{w}=\sum_i\xi_i\frac{\partial\mathbf{r}}{\partial x_i}-\sum_i\xi_i\frac{\partial\mathbf{w}(t_r)}{\partial x_i}=\sum_i\xi_i\hat{\mathbf{x}}_i-\frac{\partial\mathbf{w}}{\partial t_r}\sum_i\xi_i\frac{\partial t_r}{\partial x_i}=\boldsymbol{\xi}-\mathbf{v}\left(\boldsymbol{\xi}\cdot\nabla t_r\right). \tag{6.130}$$

Therefore,

$$-c\nabla t_r=\frac{1}{\xi}\left[\boldsymbol{\xi}-\mathbf{v}\left(\boldsymbol{\xi}\cdot\nabla t_r\right)+\boldsymbol{\xi}\times\left(\mathbf{v}\times\nabla t_r\right)\right]=\frac{1}{\xi}\left[\boldsymbol{\xi}-\left(\boldsymbol{\xi}\cdot\mathbf{v}\right)\nabla t_r\right], \tag{6.131}$$

so that

$$\nabla t_r=\frac{-\boldsymbol{\xi}}{c\xi-\boldsymbol{\xi}\cdot\mathbf{v}}\,. \tag{6.132}$$

Incorporating this result into Eq. (6.128), we have

$$\nabla\Phi=\frac{1}{4\pi\varepsilon_0}\frac{qc}{\left(\xi c-\boldsymbol{\xi}\cdot\mathbf{v}\right)^3}\left[\left(\xi c-\boldsymbol{\xi}\cdot\mathbf{v}\right)\mathbf{v}-\left(c^2-v^2+\boldsymbol{\xi}\cdot\mathbf{a}\right)\boldsymbol{\xi}\right]. \tag{6.133}$$

A similar calculation (HW#8, Problem 3) yields

$$\frac{\partial\mathbf{A}}{\partial t}=\frac{1}{4\pi\varepsilon_0}\frac{qc}{\left(\xi c-\boldsymbol{\xi}\cdot\mathbf{v}\right)^3}\left[\left(\xi c-\boldsymbol{\xi}\cdot\mathbf{v}\right)\left(-\mathbf{v}+\xi\frac{\mathbf{a}}{c}\right)+\frac{\xi}{c}\left(c^2-v^2+\boldsymbol{\xi}\cdot\mathbf{a}\right)\mathbf{v}\right]. \tag{6.134}$$

Combining these results, and introducing the vector

$$\mathbf{u}\equiv c\hat{\boldsymbol{\xi}}-\mathbf{v}=c\frac{\boldsymbol{\xi}}{\xi}-\mathbf{v}\,, \tag{6.135}$$

we obtain

$$\mathbf{E}(\mathbf{r},t)=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}=\frac{q}{4\pi\varepsilon_0}\frac{\xi}{\left(\boldsymbol{\xi}\cdot\mathbf{u}\right)^3}\left[\left(c^2-v^2\right)\mathbf{u}+\boldsymbol{\xi}\times\left(\mathbf{u}\times\mathbf{a}\right)\right]. \tag{6.136}$$

Next, we calculate the magnetic field $\mathbf{B}=\nabla\times\mathbf{A}$. We find from Eq. (6.118)

$$\nabla\times\mathbf{A}=\nabla\times\left[\frac{\mathbf{v}}{c^2}\Phi\right]=\frac{1}{c^2}\left[\Phi\left(\nabla\times\mathbf{v}\right)-\mathbf{v}\times\nabla\Phi\right]. \tag{6.137}$$

We have already calculated $\nabla\times\mathbf{v}$ (Eq. (6.125)) and $\nabla\Phi$ (Eq. (6.133)). Putting these together,

$$\nabla\times\mathbf{A} = -\frac{q}{4\pi\varepsilon_0 c}\frac{1}{\left(\boldsymbol{\xi}\cdot\mathbf{u}\right)^3}\boldsymbol{\xi}\times\left[\left(c^2-v^2\right)\mathbf{v}+\left(\boldsymbol{\xi}\cdot\mathbf{a}\right)\mathbf{v}+\left(\boldsymbol{\xi}\cdot\mathbf{u}\right)\mathbf{a}\right]. \tag{6.138}$$

The quantity in brackets is strikingly similar to the one in Eq. (6.136), which can be written, using the BAC-CAB rule, as

$$\left(c^2-v^2\right)\mathbf{u}+\boldsymbol{\xi}\times\left(\mathbf{u}\times\mathbf{a}\right)=\left(c^2-v^2\right)\mathbf{u}+\left(\boldsymbol{\xi}\cdot\mathbf{a}\right)\mathbf{u}-\left(\boldsymbol{\xi}\cdot\mathbf{u}\right)\mathbf{a}\,. \tag{6.139}$$

We see that the difference is that we have **v**'s instead of **u**'s in the first two terms. In fact, since it's all crossed into $\boldsymbol{\xi}$ anyway, we can with impunity change these **v**'s into −**u**'s; the extra term proportional to $\boldsymbol{\xi}$ disappears in the cross product. It follows that

$$\mathbf{B}(\mathbf{r},t)=\frac{1}{c}\hat{\boldsymbol{\xi}}\times\mathbf{E}(\mathbf{r},t)\,. \tag{6.140}$$

Evidently *the magnetic field of a point charge is always perpendicular to the electric field, and to the vector from the retarded point.*

The first term in **E** involving $\left(c^2-v^2\right)\mathbf{u}$ falls off as the inverse *square* of the distance from the particle. If the velocity and acceleration are both zero then $\mathbf{u}=c\hat{\boldsymbol{\xi}}$ and this term alone survives and reduces to the old electrostatic result

$$\mathbf{E}(\mathbf{r},t)=\frac{1}{4\pi\varepsilon_0}\frac{q}{\xi^2}\hat{\boldsymbol{\xi}}\,. \tag{6.141}$$

For this reason, the first term in **E** is sometimes called *the generalized Coulomb field*. (Because it does not depend on the acceleration, it is also known as the *velocity field*.) The second term (the one involving **r** × (**u** × **a**)) falls off as the inverse first power of **r** and is therefore dominant at large distances. As we will see in Section 6, it is this term that is responsible for electromagnetic radiation; accordingly, it is called the *radiation field*—or, since it is proportional to **a**, the *acceleration field*. The same terminology applies to the magnetic field.

6.7 Fields of a moving point charge moving with constant velocity

To calculate the electric and magnetic fields of a point charge moving with constant velocity, we put **a** = 0 in Eq. (6.136)

$$\mathbf{E}(\mathbf{r},t)=\frac{q}{4\pi\varepsilon_0}\frac{\left(c^2-v^2\right)\xi}{\left(\boldsymbol{\xi}\cdot\mathbf{u}\right)^3}\mathbf{u}\,. \tag{6.142}$$

In this case, using **w** = **v***t*, we find

$$\xi\mathbf{u}=c\boldsymbol{\xi}-\xi\mathbf{v}=c\left(\mathbf{r}-\mathbf{v}t_r\right)-c\left(t-t_r\right)\mathbf{v}=c\left(\mathbf{r}-\mathbf{v}t\right), \tag{6.143}$$

$$\begin{aligned}\boldsymbol{\xi}\cdot\mathbf{u}&=c\xi-\boldsymbol{\xi}\cdot\mathbf{v}=c^2\left(t-t_r\right)-\left(\mathbf{r}-\mathbf{v}t_r\right)\cdot\mathbf{v}=c^2\left(t-t_r\right)-\mathbf{r}\cdot\mathbf{v}+v^2t_r=\\&=c^2t-\mathbf{r}\cdot\mathbf{v}-\left(c^2-v^2\right)t_r=\sqrt{\left(c^2t-\mathbf{r}\cdot\mathbf{v}\right)^2+\left(c^2-v^2\right)\left(r^2-c^2t^2\right)},\end{aligned} \tag{6.144}$$

where we used Eq. (6.110) with the negative sign in front of the radical there. Eq. (6.144) is consistent with Eq. (6.112).

In HW#8 (Problem 2) shows that the radical in Eq. (6.112) could be written as

$$Rc\sqrt{1-\frac{v^2}{c^2}\sin^2\theta}\ , \tag{6.145}$$

where $\mathbf{R} \equiv \mathbf{r} - \mathbf{v}t$ is the vector from the *present* position of the particle to the field point $\mathbf{r}$ and θ is the angle between $\mathbf{R}$ and $\mathbf{v}$ (Fig. 6.4). Thus,

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0}\frac{\left(1-\frac{v^2}{c^2}\right)}{\left(1-\frac{v^2}{c^2}\sin^2\theta\right)^{3/2}}\frac{\mathbf{R}}{R^3}\ . \tag{6.146}$$

Notice that $\mathbf{E}$ points along the line from the *present* position of the particle. This is an extraordinary coincidence, since the "message" came from the *retarded* position. Because of the $\sin^2\theta$ in the denominator, the field of a fast-moving charge is flattened out like a pancake in the direction perpendicular to the motion (Fig. 6.5). In the forward and backward directions $\mathbf{E}$ is *reduced* by a factor $1-v^2/c^2$ relative to the field of a charge at rest; in the perpendicular direction it is *enhanced* by a factor $\sqrt{1-v^2/c^2}$.

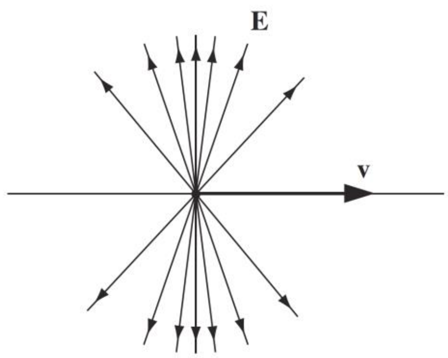

Fig. 6.5

As for $\mathbf{B}$, we have

$$\hat{\boldsymbol{\xi}} = \frac{\mathbf{r}-\mathbf{v}t_r}{\xi} = \frac{(\mathbf{r}-\mathbf{v}t)+(t-t_r)\mathbf{v}}{\xi} = \frac{\mathbf{R}}{\xi} + \frac{\mathbf{v}}{c}\ . \tag{6.147}$$

and therefore

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{c}\hat{\boldsymbol{\xi}} \times \mathbf{E}(\mathbf{r},t) = \frac{1}{c^2}\mathbf{v} \times \mathbf{E}(\mathbf{r},t)\ . \tag{6.148}$$

Lines of $\mathbf{B}$ *circle around* the charge, as shown in Fig. 6.6.

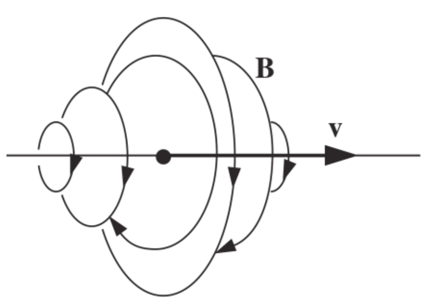

Fig. 6.6

When $v \ll c$, the fields of a point charge moving at constant velocity are reduced to

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0}\frac{\mathbf{R}}{R^3}; \quad \mathbf{B}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\frac{q}{R^3}\mathbf{v}\times\mathbf{R}. \tag{6.149}$$

The first is essentially Coulomb's law, and the second is "Biot-Savart's law" for a point charge.