Section 6: Magnetostatics

Magnetic Fields in Matter

In the previous sections we assumed that the current density **J** is a known function of coordinates. In the presence of matter this is not always true. The atoms in matter have electrons that give rise to effective atomic currents, the current density of which is a rapidly fluctuating quantity. Only its average over a macroscopic volume is known. Furthermore, the atomic electrons contribute to intrinsic magnetic moments in addition to those resulting from their orbital motion. All these moments can give rise to dipole fields that vary appreciably on the atomic scale.

The process of averaging these microscopic quantities results in the macroscopic description of magnetic matter. In particular, a macroscopic magnetic field **B** obeys the same equation

$$\nabla \cdot \mathbf{B} = 0 \tag{6.1}$$

as its microscopic analog. Then we can still use the concept of a vector potential **A** whose curl gives **B**.

In the presence of a magnetic field, matter becomes *magnetized*; that is the dipoles acquire a net alignment along some direction. There are two mechanisms that account for this magnetic polarization:

(1) *paramagnetism*: the dipoles associated with the spins of unpaired electrons experience a torque tending to line them up parallel to the field;

(2) *diamagnetism*: the orbital speed of the electrons is altered in such a way as to change the orbital dipole moment in a direction opposite to the field.

Whatever the *cause,* we describe the state of magnetic polarization by the vector quantity

M = *magnetic dipole moment per unit volume*.

M is called the *magnetization*; it plays a role analogous to the polarization **P** in electrostatics. Note that *ferromagnetic* materials are magnetized in the *absence* of applied field. Below we will not worry about how the magnetization *got* there – it could be paramagnetism, diamagnetism, or even ferromagnetism – we shall take **M** as *given,* and calculate the field this magnetization itself produces.

Suppose we have a piece of magnetized material; the magnetic dipole moment per unit volume, **M**, is given. What field does this object produce? The vector potential of a single dipole **m** is given by $\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{|\mathbf{r}|^3}$. In the magnetized object, each volume element d^3r' carries a dipole moment $\mathbf{M}(\mathbf{r}')d^3r'$ (Fig. 6.1), so the total vector potential is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{M}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} d^3r' \ . \tag{6.2}$$

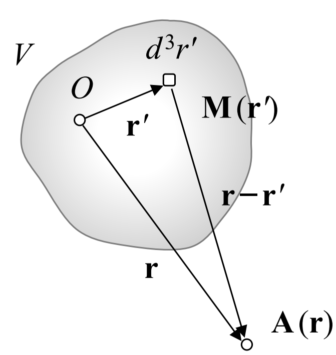


Fig. 6.1

Using the identity

$$\nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \ , \tag{6.3}$$

Eq. (6.2) can be written in form

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \mathbf{M}(\mathbf{r}')\times\nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|} d^3r' \ . \tag{6.4}$$

Integrating by parts we obtain:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left\{\int \frac{1}{|\mathbf{r}-\mathbf{r}'|}\nabla'\times\mathbf{M}(\mathbf{r}')d^3r' - \int \nabla'\times\frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r'\right\} , \tag{6.5}$$

where we used the identity

$$\nabla\times(\psi\mathbf{a}) = \psi(\nabla\times\mathbf{a}) + (\nabla\psi)\times\mathbf{a} \ . \tag{6.6}$$

Now we rewrite the second integral in Eq. (6.5) using the divergence theorem. For an arbitrary vector field $\mathbf{v}(\mathbf{r})$ and a constant vector $\mathbf{c}$ the divergence theorem gives

$$\int_V \nabla\cdot(\mathbf{v}\times\mathbf{c})d^3r = \oint_S (\mathbf{v}\times\mathbf{c})\cdot\mathbf{n}da \ . \tag{6.7}$$

On the other hand, using the cyclic permutation in the triple product, we have

$$\nabla\cdot(\mathbf{v}\times\mathbf{c}) = \mathbf{c}\cdot(\nabla\times\mathbf{v}) - \mathbf{v}\cdot(\nabla\times\mathbf{c}) = \mathbf{c}\cdot(\nabla\times\mathbf{v}) , \tag{6.8}$$

and

$$(\mathbf{v}\times\mathbf{c})\cdot\mathbf{n} = -\mathbf{c}\cdot(\mathbf{v}\times\mathbf{n}) \ . \tag{6.9}$$

Therefore, Eq. (6.7) can be written as

$$\int_V \mathbf{c}\cdot(\nabla\times\mathbf{v})d^3r = -\oint_S \mathbf{c}\cdot(\mathbf{v}\times\mathbf{n})da \ . \tag{6.10}$$

Since $\mathbf{c}$ is arbitrary, it follows from Eq. (6.10) that

$$\int_V (\nabla\times\mathbf{v})d^3r = -\oint_S (\mathbf{v}\times\mathbf{n})da \ . \tag{6.11}$$

Therefore, Eq. (6.5) yields

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left\{\int \frac{\nabla'\times\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r' + \oint \frac{\mathbf{M}(\mathbf{r}')\times\mathbf{n}'}{|\mathbf{r}-\mathbf{r}'|}da'\right\} . \tag{6.12}$$

The first term looks like the potential of a *volume* current density,

$$\mathbf{J}_M = \nabla\times\mathbf{M} , \tag{6.13}$$

while the second term looks like the potential of a *surfac*e current density

$$\mathbf{K}_M = \mathbf{M}\times\mathbf{n} , \tag{6.14}$$

where $\mathbf{n}$ is the normal unit vector. With these definitions

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left\{\int \frac{\mathbf{J}_M(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r' + \oint \frac{\mathbf{K}_M(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}da'\right\} . \tag{6.15}$$

This means that the potential (and hence also the field) of a magnetized object is the same as would be produced by a volume current $\mathbf{J}_M = \nabla\times\mathbf{M}$ throughout the material, plus a surface current $\mathbf{K}_M = \mathbf{M}\times\mathbf{n}$, on the boundary. Instead of integrating the contributions of all the infinitesimal dipoles, as in Eq. (6.2), we first determine these *bound* or *magnetization currents*, and then find the field they produce, in the same way we would calculate the field of any other volume and surface currents. Notice the striking parallel with the electrostatics: there the field of a polarized object was the same as that of a polarization volume charge $\rho_P = -\nabla \cdot \mathbf{P}$ plus a polarization surface charge $\sigma_P = \mathbf{P} \cdot \mathbf{n}$.

Now, we discuss the physical picture for bound currents. Fig. 6.2 shows a thin slab of uniformly magnetized material, with dipoles represented by tiny current loops. Notice that if the magnetization is uniform all the "internal" currents cancel. However, at the edge there is no adjacent loop to do the canceling. The whole magnetized slab is therefore equivalent to a single ribbon of current I flowing around the boundary (Fig. 6.2).

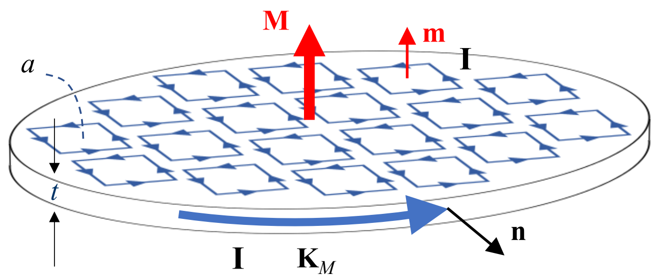


Fig. 6.2

What is this current, in terms of **M**? Say that each of the tiny loops has area a and thickness t. In terms of the magnetization M, its dipole moment is $m = Mat$. In terms of the circulating current I, however, $m = Ia$. Therefore $I = Mt$, so the surface current is $K_M = I/t = M$. Using the outward-drawn unit vector **n** (Fig. 6.2), the direction of K_M is conveniently indicated by the cross product: $\mathbf{K}_M = \mathbf{M} \times \mathbf{n}$. This expression also records the fact that there is no current on the top or bottom surface of the slab; here **M** is parallel to **n**, so the cross product vanishes.

This bound surface current is exactly what we obtained above [Eq. (6.14)]. It is a peculiar kind of current, in the sense that no single charge makes the whole trip – on the contrary, each charge moves only in a tiny little loop within a single atom. Nevertheless, the net effect is a macroscopic current flowing over the surface of the magnetized object.

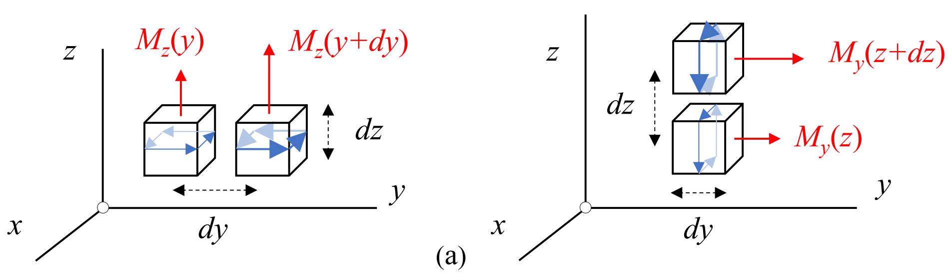


Fig. 6.3

When the magnetization is nonuniform, the internal currents no longer cancel. Fig. 6.3a shows two adjacent chunks of magnetized material, with a larger arrow on the one to the right suggesting greater magnetization at that point. On the surface where they join, there is a net current in the x-direction,

$$I_x = \left[M_z(y+dy) - M_z(y)\right]dz = \frac{\partial M_z}{\partial y} dydz \ , \tag{6.16}$$

The corresponding volume current density is therefore

$$\left(J_M\right)_x = \frac{\partial M_z}{\partial y} \ . \tag{6.17}$$

By the same token, a nonuniform magnetization in the y-direction would contribute an amount $-\partial M_y / \partial z$ (Fig. 6.3b), so

$$\left(J_M\right)_x = \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z} \ . \tag{6.18}$$

In general, then, $\mathbf{J}_M = \nabla \times \mathbf{M}$, which is consistent, again, with the result (6.13). Incidentally, like any other steady current, $\mathbf{J}_M$ obeys the conservation law:

$$\nabla \cdot \mathbf{J}_M = 0 \ , \tag{6.19}$$

because the divergence of a curl is always zero.

For magnetization $\mathbf{M}$ localized in space in the presence of free currents $\mathbf{J}$, the expression for the vector potential has the form

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') + \nabla' \times \mathbf{M}(\mathbf{r}')}{\left|\mathbf{r} - \mathbf{r}'\right|} d^3r' \ , \tag{6.20}$$

where the integral is taken over all space. This is because the magnetization is zero at infinity.

We note that the surface magnetization currents are included in Eq. (6.20) implicitly, due to an abrupt change of $\mathbf{M}$ at the boundary of the magnetized material. According to Eq. (6.11), we have $\int_V \left(\nabla \times \mathbf{M}\right) d^3r = -\oint_S \left(\mathbf{M} \times \mathbf{n}\right) da$. Applying this relation to a small volume of a magnetized medium with an abrupt change of magnetization at the boundary from $\mathbf{M}$ to zero, we find $\left(\nabla \times \mathbf{M}\right) d^3r = \left(\mathbf{M} \times \mathbf{n}\right) da$, where $\mathbf{n}$ is the normal to that surface.

Eq. (6.20) implies that taking into account the bound currents (magnetization) leads to the effective current density $\mathbf{J} + \nabla \times \mathbf{M}$ and consequently to a new macroscopic equation for the magnetic field:

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \nabla \times \mathbf{M}\right) \ . \tag{6.21}$$

The term $\nabla \times \mathbf{M}$ can be combined with $\mathbf{B}$ to define a new macroscopic field $\mathbf{H}$,

$$\mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \ . \tag{6.22}$$

Therefore, the macroscopic equations become

$$\nabla \times \mathbf{H} = \mathbf{J} \ , \tag{6.23}$$

$$\nabla \cdot \mathbf{B} = 0 \ . \tag{6.24}$$

Eq. (6.23) is the Ampere's law for magnetostatics with magnetized materials. It represents a convenient way to find magnetic field $\mathbf{H}$ using only free currents. In the integral form of the Ampere's law reads

$$\oint \mathbf{H} \cdot d\mathbf{l} = I \ , \tag{6.25}$$

where I is the electric free current passing through the loop.

To complete the description of macroscopic magnetostatics, there must be a *constitutive* relation between $\mathbf{H}$ and $\mathbf{B}$. For diamagnetic and paramagnetic materials and not too strong fields it is customary to write the relation between the magnetization and magnetic field in the form

$$\mathbf{M} = \chi_m \mathbf{H} \ . \tag{6.26}$$

Note that **H** rather than **B** enters this equation. The constant χ_m is called the *magnetic susceptibility*. It is dimensionless quantity which is positive for paramagnets and negative for diamagnets. Typical values are around 10^{-5}.

Materials that obey Eq. (6.26) are called linear media. In view of Eq. (6.22) for linear media

$$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}) = \mu_0 (1 + \chi_m)\mathbf{H} \ . \tag{6.27}$$

Thus, **B** is also proportional to **H**:

$$\mathbf{B} = \mu \mathbf{H} \ , \tag{6.28}$$

where $\mu = \mu_0 (1 + \chi_m)$ is the *magnetic permeability* of the material.

For the ferromagnetic substances, Eq. (6.28) must be replaced by a nonlinear functional relationship,

$$\mathbf{B} = \mathbf{F}(\mathbf{H}) \ . \tag{6.29}$$

Function **F**(**H**) depends on the history of preparation of the material which leads to the phenomenon of hysteresis. The incremental permeability μ (**H**) is defined as the derivative of B with respect to H. For high-permeability substances, the relative incremental permeability μ/μ_0 can be as high as 10^6.

Boundary Conditions

Just as the electric field suffers a discontinuity at a surface *charge*, so the magnetic field is discontinuous at a surface *current*. Only this time it is the tangential component that changes. Indeed, if we apply $\nabla \cdot \mathbf{B} = 0$ in the integral form

$$\oint_S \mathbf{B} \cdot \mathbf{n} da = 0 \tag{6.30}$$

to a thin gaussian pillbox straddling the surface (Fig. 6.4), we obtain

$$B_2^{\perp} - B_1^{\perp} = 0 \ , \tag{6.31}$$

where $B^{\perp}$ is the component of the magnetic field **B** perpendicular to the surface. Eq. (6.31) tells us that $B^{\perp}$ is continuous at the interface. The perpendicular component of **H** is however discontinuous if the magnetization of the two media are different:

$$H_2^{\perp} - H_1^{\perp} = -\left(M_2^{\perp} - M_1^{\perp}\right) \ . \tag{6.32}$$

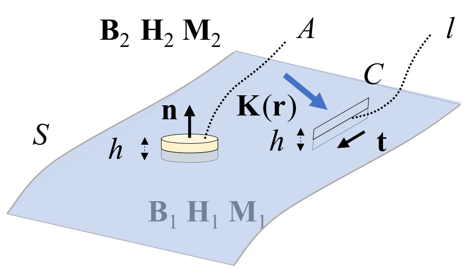


Fig. 6.4

As for the tangential components, from Ampere's law $\nabla \times \mathbf{H} = \mathbf{J}$ an amperian loop running perpendicular to the current (Fig. 6.5) yields

$$\int_S \nabla \times \mathbf{H} \cdot \mathbf{n} da = \oint_C \mathbf{H} \cdot d\mathbf{l} = \left(H_2^{\parallel} - H_1^{\parallel}\right) l = Kl \tag{6.33}$$

or

$$H_2^{\parallel} - H_1^{\parallel} = K \ . \tag{6.34}$$

Thus, the component of **H** that is parallel to the surface but perpendicular to the current is discontinuous in the amount *K*. A similar amperian loop running parallel to the current reveals that the parallel component is continuous. These results can be summarized in a single formula:

$$\mathbf{H}_2^{\parallel} - \mathbf{H}_1^{\parallel} = (\mathbf{K} \times \mathbf{n}) \ , \tag{6.35}$$

where **n** is a unit vector perpendicular to the surface, pointing "upward." Vector-multiplying Eq. (6.35) by $\mathbf{n}$ from left, this result can be equivalently written as

$$\mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K} . \tag{6.36}$$

This follows from the fact that $\mathbf{n} \times \mathbf{H}_{1,2}^{\parallel} = \mathbf{n} \times \mathbf{H}_{1,2}$ and $\mathbf{n} \times (\mathbf{K} \times \mathbf{n}) = \mathbf{K}(\mathbf{n} \cdot \mathbf{n}) - \mathbf{n}(\mathbf{n} \cdot \mathbf{K}) = \mathbf{K}$.

Boundary Value Problems in Magnetostatics

The basic equations of magnetostatics are

$$\nabla \cdot \mathbf{B} = 0 \ , \tag{6.37}$$

$$\nabla \times \mathbf{H} = \mathbf{J} \ , \tag{6.38}$$

with some constitutive relation between **B** and **H** such as Eq. (6.28) or (6.29). Since the divergence of **B** is zero, we can always introduce a vector potential **A** such that

$$\mathbf{B} = \nabla \times \mathbf{A} \ . \tag{6.39}$$

In general case, when the relationship between **H** and **B** is non-linear, the second equation (6.38) becomes very complicated even if the current distribution is simple. For linear media with $\mathbf{B} = \mu \mathbf{H}$, the equation takes the form

$$\nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A} \right) = \mathbf{J} \ . \tag{6.40}$$

If μ is constant over a finite region in space, in can be taken out of differentiation and Eq. (6.40) becomes

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu \mathbf{J} \ . \tag{6.41}$$

With the choice of gauge

$$\nabla \cdot \mathbf{A} = 0 \tag{6.42}$$

this turns into a Poisson's equation

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J} \ , \tag{6.43}$$

similar to what we had in vacuum with a modified current density $\mu \mathbf{J} / \mu_0$. The situation is analogous the treatment of uniform dielectric media where the effective charge density in Poisson's equation is $\varepsilon_0 \rho / \varepsilon$. Solutions of Eq. (6.43) in different linear media must be matched across the boundary surfaces using the boundary conditions.

Magnetic Scalar Potential

If the current density vanishes in some finite region in space, i.e. $\mathbf{J} = 0$, Eq. (6.38) becomes

$$\nabla \times \mathbf{H} = 0 \ . \tag{6.44}$$

This implies that we can introduce a magnetic scalar potential Φ_M such that

$$\mathbf{H} = -\nabla \Phi_M \ , \tag{6.45}$$

just as $\mathbf{E} = -\nabla\Phi$ in the electrostatics. Assuming that the medium is *linear* [i.e. described by Eq. (6.28)] and *uniform* (i.e. the magnetic permeability is constant in space), Eq. (6.37) together with Eq. (6.45) lead to the Laplace equation for the magnetic scalar potential:

$$\nabla^2 \Phi_M = 0 \ . \tag{6.46}$$

Therefore, one can use methods of solving differential equations to find the magnetic scalar potential and therefore the magnetic fields **H** and **B**.

In *hard ferromagnets* the situation becomes simpler. In this case the magnetization **M** is largely independent of the magnetic field and therefore we can assume that **M** is a given function of coordinates. We can exploit equations $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$ and $\nabla \cdot \mathbf{B} = 0$ to obtain

$$\nabla \cdot \mathbf{B} = \mu_0 \nabla \cdot \left(\mathbf{H} + \mathbf{M}\right) = 0 \ , \tag{6.47}$$

and hence

$$\nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M} \ . \tag{6.48}$$

Now using the magnetic scalar potential (6.45) we obtain a magnetostatic Poisson equation:

$$\nabla^2 \Phi_M = -\rho_M \ , \tag{6.49}$$

where the *effective magnetic charge density* is given by

$$\rho_M = -\nabla \cdot \mathbf{M} \ . \tag{6.50}$$

The solution for the potential Φ_M , if there are no boundary surfaces, is

$$\Phi_M(\mathbf{r}) = \frac{1}{4\pi} \int \frac{\rho_M(\mathbf{r}')}{\left|\mathbf{r} - \mathbf{r}'\right|} d^3 r' = -\frac{1}{4\pi} \int \frac{\nabla' \cdot \mathbf{M}(\mathbf{r}')}{\left|\mathbf{r} - \mathbf{r}'\right|} d^3 r' \ . \tag{6.51}$$

In solving magnetostatics problems with a given magnetization distribution which changes abruptly at the boundaries of the specimen it is convenient to introduce the *magnetic surface charge density*. If a hard ferromagnet has volume *V* and surface *S*, we specify **M**(**r**) inside *V* and assume that it falls suddenly to zero at the surface *S*. Application of the divergence theorem to ρ_M (6.50) in a Gaussian pillbox straddling the surface shows that the effective magnetic surface change density is given by

$$\sigma_M = \mathbf{n} \cdot \mathbf{M} \ , \tag{6.52}$$

where **n** is the outwardly directed normal. Then instead of (6.51) the potential is represented as follows

$$\Phi_M(\mathbf{r}) = -\frac{1}{4\pi} \int_V \frac{\nabla' \cdot \mathbf{M}(\mathbf{r}')}{\left|\mathbf{r} - \mathbf{r}'\right|} d^3 r' + \frac{1}{4\pi} \oint_S \frac{\mathbf{n}' \cdot \mathbf{M}(\mathbf{r}')}{\left|\mathbf{r} - \mathbf{r}'\right|} da' \ . \tag{6.53}$$

An important special case it that of uniform magnetization throughout the volume *V*. Then the first term vanishes; only the surface integral over σ_M contributes.

Examples of Boundary-Value Problems in Magnetostatics

To illustrate different methods for the solution of boundary value problems in magnetostatics, we consider two examples.

Example 1 is an infinite slab of magnetic material which has a uniform magnetization **M** oriented either parallel (Fig. 6.5a) or perpendicular (Fig. 6.5b) to the surfaces of the slab. We need to calculate the magnetic fields **H** and **B** everywhere in space.

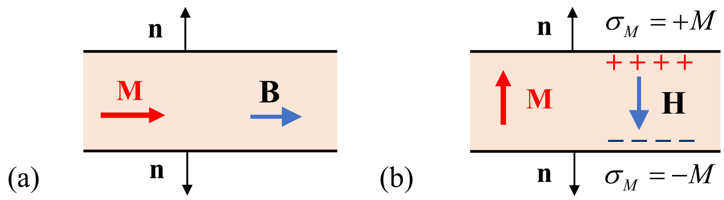


Fig. 6.5

Since electric current $\mathbf{J}=0$, $\nabla\times\mathbf{H}=0$ and $\nabla\cdot\mathbf{H}=-\nabla\cdot\mathbf{M}$. This implies that $\rho_M=-\nabla\cdot\mathbf{M}$ plays a role of magnetic charge density, and **H** can be found like electric field **E** in electrostatics. In case (a), since **M**= const, $\rho_M=-\nabla\cdot\mathbf{M}=0$, and therefore $\mathbf{H}=0$ everywhere in space. Therefore $\mathbf{B}=0$ outside the slab and $\mathbf{B}=\mu_0\mathbf{M}$ inside the slab. In case (b) magnetization creates positive surface charge, $\sigma_M=+M$, on the top surface and negative surface charge, $\sigma_M=-M$, on the bottom surface. These charges generate magnetic field, $\mathbf{H}=-\mathbf{M}$, opposite to the magnetization within the slab and no field outside, $\mathbf{H}=0$. This makes field **B** zero everywhere in space.

Example 2 is a sphere of radius *R*, with a uniform permanent magnetization **M** parallel to the *z* axis (Fig. 6.6). The simplest way to solve this problem is to use the notion of magnetization charge and the magnetic scalar potential. The problem reduces to finding the potential for a specified change density $\sigma_M(\theta)=M\cos\theta$ glued over the surface of a spherical shell of radius *R*. We need to find the resulting potential inside and outside the sphere.

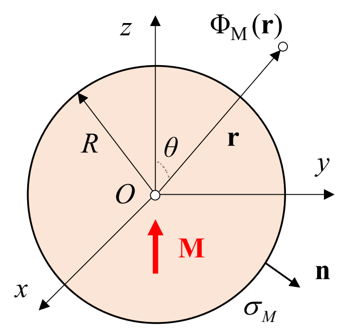


Fig. 6.6

Taking into account that the potential is finite at infinity and at the center of the sphere, we have for the interior region

$$\Phi_{in}(r,\theta)=\sum_{l=0}^{\infty}A_l r^l P_l(\cos\theta),\ \ r\le R\ , \tag{6.54}$$

and for the exterior region

$$\Phi_{out}(r,\theta)=\sum_{l=0}^{\infty}\frac{C_l}{r^{l+1}}P_l(\cos\theta),\ \ r\ge R\ . \tag{6.55}$$

These two functions must be joined together by the appropriate boundary conditions at the surface itself. Since the boundary conditions at $r = R$ we obtain that

$$\sum_{l=0}^{\infty}A_l R^l P_l(\cos\theta)=\sum_{l=0}^{\infty}\frac{C_l}{R^{l+1}}P_l(\cos\theta)\ . \tag{6.56}$$

It follows from here that

$$C_l = A_l R^{2l+1}\ . \tag{6.57}$$

The normal derivative of the potential suffers a discontinuity at the surface so that

$$\left(\frac{\partial\Phi_{out}}{\partial r}-\frac{\partial\Phi_{in}}{\partial r}\right)_{r=R}=-\sigma_M(\theta)\ . \tag{6.58}$$

Thus

$$-\sum_{l=0}^{\infty}(l+1)\frac{C_l}{R^{l+2}}P_l(\cos\theta)-\sum_{l=0}^{\infty}lA_lR^{l-1}P_l(\cos\theta)=-M\cos\theta\ , \tag{6.59}$$

or using Eq. (6.57):

$$\sum_{l=0}^{\infty}(2l+1)A_lR^{l-1}P_l(\cos\theta)=M\cos\theta\ . \tag{6.60}$$

It follows from here that the only term which survives is the one with $l = 1$ so that

$$A_1 = \frac{M}{3} \ , \tag{6.61}$$

and therefore

$$C_1 = \frac{M}{3} R^3 \ . \tag{6.62}$$

According to Eq. (6.54) the solution inside the sphere, $r \le R$, is

$$\Phi_M(r,\theta) = \frac{M}{3} r\cos\theta = \frac{M}{3} z \ . \tag{6.63}$$

This implies that the magnetic field **H** is

$$\mathbf{H} = -\nabla\Phi_M = -\frac{M}{3}\hat{\mathbf{z}} = -\frac{\mathbf{M}}{3} \tag{6.64}$$

and the magnetic field **B** is

$$\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M}) = \frac{2\mu_0\mathbf{M}}{3} \ . \tag{6.65}$$

Note that **B** is parallel to **M** while **H** is antiparallel to **M** inside the sphere.

Outside the sphere, $r \ge R$, according to Eq. (6.55) the potential is

$$\Phi_M(r,\theta) = \frac{M_0}{3}\frac{R^3}{r^2}\cos\theta, \quad r \ge R \ . \tag{6.66}$$

This is the potential which is produced by a point dipole

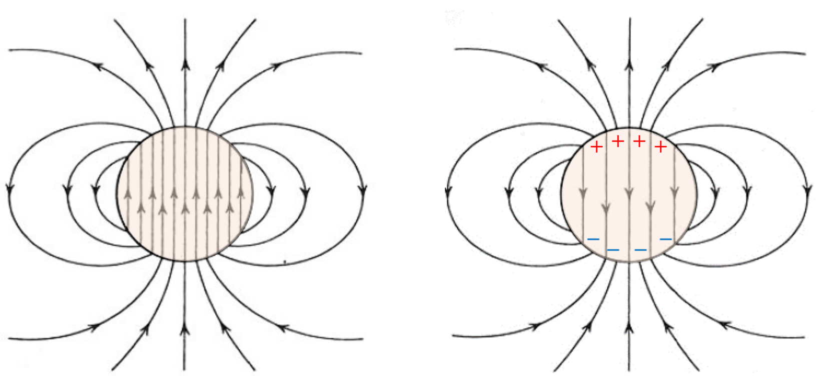
$$\Phi_M(r,\theta) = \frac{1}{4\pi}\frac{\mathbf{m}\cdot\mathbf{r}}{r^3} \ , \tag{6.67}$$

with the dipole moment

$$\mathbf{m} = \frac{4\pi R^3}{3}\mathbf{M} \ . \tag{6.68}$$

We see that for the *sphere* with uniform magnetization, the fields are not only dipole in character asymptotically, but also close to the sphere. For this special geometry, there are no higher multipoles.

The lines of **B** and **H** are shown in Fig. 6.7. The lines of **B** are continuous closed paths, but those of **H** terminate on the surface because there is an effective surface-charge density σ_M.



B field lines　　**H** field lines　　**Fig. 6.7**

Magnetized Sphere in an External Field; Permanent Magnets

Now we consider a problem of a uniformly magnetized sphere in an external magnetic field. We can use results of the preceding section, because of the linearity of the field equations, which allows us to superpose a uniform magnetic induction $\mathbf{B}_0 = \mu_0 \mathbf{H}_0$ throughout all space. From Eqs. (6.64) and (6.65) we find that the magnetic fields inside the sphere are now

$$\mathbf{B} = \mathbf{B}_0 + \frac{2\mu_0 \mathbf{M}}{3} , \tag{6.69}$$

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B}_0 - \frac{1}{3}\mathbf{M} . \tag{6.70}$$

Now imagine that the sphere is not a permanently magnetized object, but rather a paramagnetic or diamagnetic substance of permeability μ. Then the magnetization **M** is a result of the application of the external field. To find the magnitude of **M** we use Eq. (6.28):

$$\mathbf{B} = \mu \mathbf{H} , \tag{6.71}$$

so that

$$\mathbf{B}_0 + \frac{2\mu_0 \mathbf{M}}{3} = \mu\left(\frac{1}{\mu_0}\mathbf{B}_0 - \frac{1}{3}\mathbf{M}\right) . \tag{6.72}$$

This gives a magnetization

$$\mathbf{M} = \frac{3}{\mu_0}\left(\frac{\mu - \mu_0}{\mu + 2\mu_0}\right)\mathbf{B}_0 . \tag{6.73}$$

For a ferromagnetic substance, the arguments of the preceding paragraph fail. Eq. (6.73) implies that the magnetization vanishes when the external field vanishes. The existence of permanent magnets contradicts this result. The nonlinear relation **B** = **F**(**H**) and the phenomenon of hysteresis allow the creation of permanent magnets. We can solve Eqs. (6.69), (6.70) for one relation between **H** and **B** by eliminating **M**:

$$\mathbf{B} = -2\mu_0 \mathbf{H} + 3\mathbf{B}_0 . \tag{6.74}$$

The hysteresis curve provides the other relation between **B** and **H**, so that specific values can be found for any external field. Eq. (6.74) corresponds to a line with slope –2 on the hysteresis diagram with intercept $3B_0$ on the *y* axis, as in Fig. 6.8. Suppose, for example, that the external field is increased until the ferromagnetic sphere becomes saturated and then decreased to zero. The internal B and H will then be given by the point marked P in Fig. 6.9. The magnetization can then be found from Eq. (6.69) or Eq. (6.70) with $B_0 = 0$.

The relation (6.74) between **B** and **H** is specific to the sphere. For other geometries other relations pertain.

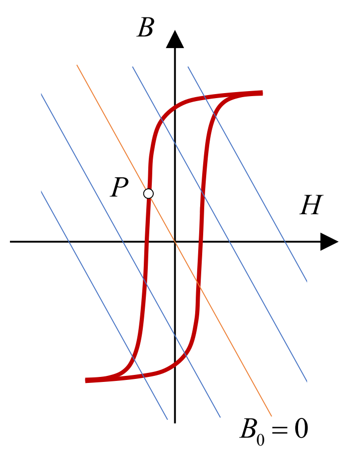


Fig. 6.8