Section 4: Electrostatics of Dielectrics

Dielectrics and Polarizability

There are two large classes of substances: conductors and insulators (or dielectrics). In contrast to metals where charges are free to move throughout the material, in dielectrics all the charges are attached to specific atoms and molecules. These charges are known as *bound* charges. These charges are able, however, to be displaced within an atom or a molecule. Such microscopic displacements are not as dramatic as the rearrangement of charge in a conductor, but their cumulative effects account for the characteristic behavior of dielectric materials.

When an external electric field is applied to a dielectric material this material becomes *polarized*, which means that acquires a *dipole moment*. This property of dielectrics is known as *polarizability*. Basically, polarizability is a consequence of the fact that molecules, which are the building blocks of all substances, are composed of both positive charges (nuclei) and negative charges (electrons). When an electric field acts on a molecule, the positive charges are displaced along the field, while the negative charges are displaced in a direction opposite to that of the field. The effect is therefore to pull the opposite charges apart, i.e., to polarize the molecule.

It is convenient to define *the polarizability* α of an atom in terms of the local electric field at the atom:

$$\mathbf{p} = \alpha \mathbf{E}_{loc}, \tag{4.1}$$

where $\mathbf{p}$ is the dipole moment. For a non-spherical atom α will be a tensor.

There are different types of polarization processes, depending on the structure of the molecules which constitute the solid. If the molecule has a *permanent moment,* i.e., a moment is present even in the absence of an electric field, we speak of a *dipolar* molecule, and a *dipolar* substance.

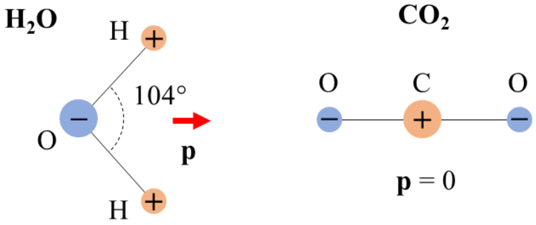

Fig. 4.1 (a) The water molecule (left) and CO_2 molecule (right).

An example of a dipolar molecule is the H_2O molecule in Fig. 4.1. The dipole moments of the two OH bonds add vectorially to give a nonvanishing net dipole moment. Some molecules are nondipolar, possessing no permanent moments; a common example is the CO_2 molecule in Fig. 4.1. The moments of the two CO bonds cancel each other because of the rectilinear shape of the molecule, resulting in a zero net dipole moment in the absence of electric field.

Despite the fact that the individual molecules in a dipolar substance have permanent moments, the net polarization vanishes in the absence of an external field because the molecular moments are randomly oriented, resulting in a complete cancellation of the polarization. When a field is applied to the substance, however, the molecular dipoles tend to align with the field. The reason is that the energy of a dipole $\mathbf{p}$ in a local external field $\mathbf{E}_{loc}$ is $U = -\mathbf{p} \cdot \mathbf{E}_{loc}$. It has a minimum when the dipole is parallel to the field. This results in a net non-vanishing dipole moment of the material. This mechanism for polarizability is called *dipolar polarizability*.

If the molecule contains ionic bonds, then the field tends to stretch the lengths of these bonds. This occurs in NaCl, for instance, because the field tends to displace the positive ion Na^+ to the right (see Fig.4.2), and the negative ion Cl^- to the left, resulting in a stretching in the length of the bond. The effect of this change in length is to produce a net dipole moment in the unit cell where previously there was none. Since the polarization here is due to the relative displacements of oppositely charged ions, we speak of *ionic polarizability*.

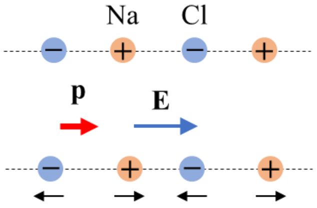

Fig. 4.2 Ionic polarization in NaCl. The field displaces Na^+ and Cl^- ions in opposite directions, changing the bond length.

Ionic polarizability exists whenever the substance is either ionic, as in NaCl, or dipolar, as in H_2O, because in each of these classes there are ionic bonds present. But in substances in which such bonds are missing, such as Si and Ge, ionic polarizability is absent.

The third type of polarizability arises because the individual ions or atoms in a molecule are themselves polarized by the field. In the case of NaCl, each of the Na^+ and Cl^- ions are polarized. Thus, the Na^+ ion is polarized because the electrons in its various shells are displaced to the left relative to the nucleus, as shown in Fig. 4.3. We are clearly speaking here *of electronic polarizability*.

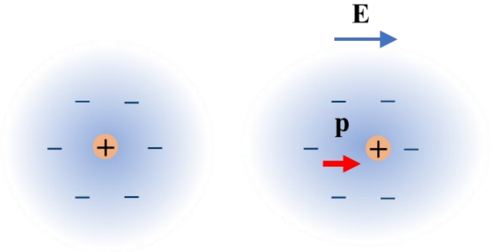

Fig. 4.3 Electronic polarization: Unpolarized atom (left) and atom polarized in the external field (right).

Electronic polarizability arises even in the case of a neutral atom, again because of the relative displacement of the orbital electrons.

In general, therefore, the total polarizability is given by

$$\alpha = \alpha_e + \alpha_i + \alpha_d , \qquad (4.2)$$

which is the sum of the electronic, ionic, and dipolar polarizabilities, respectively. The electronic contribution is present in any type of substance, but the presence of the other two terms depends on the material under consideration.

The relative magnitudes of the various contributions in Eq. (4.2) are such that in nondipolar, ionic substances the electronic part is often of the same order as the ionic. In dipolar substances, however, the greatest contribution comes from the dipolar part. This is the case for water, for example.

Polarization

If an electric field is applied to a medium made up of large number of atoms or molecules, the charges bound in each molecule respond to the applied field resulting in the redistribution of charges and leading to a *polarization* of the medium. The electric polarization $\mathbf{P}(\mathbf{r})$ is defined as the dipole moment per unit volume. The polarization is a macroscopic quantity because it involves averaging of the dipole moments over a volume which contains many dipoles. We assume that the response of the system to an applied field is linear. This excludes ferroelectricity from discussion, but otherwise is no real restriction provided the field strengths do not become extremely large. As a further simplification we suppose that the *medium is* *isotropic*. Then the induced polarization **P** is parallel to **E** with a coefficient of proportionality that is independent of direction:

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E} . \tag{4.3}$$

The constant χ_e is the *electric susceptibility* of the medium.

An important point to note that the electric field which enters Eq. (4.3) is the a macroscopic electric field which is different from a local electric field entering Eq. (4.1). The macroscopic field is the average over volume with a size being large compared to an atomic size.

Now, we look at the medium from a macroscopic point of view assuming that the medium contains free charges characterized by the charge density ρ and bound charges characterized by polarization **P**. We can build up the potential and the field by linear superposition of the contributions from each macroscopically small volume element δV at the variable point $\mathbf{r}'$. The free charge contained in volume δV is $\rho(\mathbf{r}')\ \delta V$ and the dipole moment of δV is $\mathbf{P}(\mathbf{r}')\delta V$. If there are no higher macroscopic multipole moment densities, the contribution to the potential $\delta\Phi(\mathbf{r},\mathbf{r}')$ caused by the configuration of moments in δV is given without approximation by

$$\delta\Phi(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi\varepsilon_0}\left[\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\delta V + \frac{\mathbf{P}(\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\delta V\right]. \tag{4.4}$$

provided **r** is outside δV. The first term is the contribution from free charges and the second term is due to a volume distribution of dipoles. We now treat δV as (macroscopically) infinitesimal, put it equal to d^3r', and integrate over the volume of the dielectric to obtain the potential

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \left[\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \frac{\mathbf{P}(\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\right] d^3r' . \tag{4.5}$$

To simplify this equation, we use the identity

$$\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = -\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|} , \tag{4.6}$$

where ∇' implies differentiation with respect to $\mathbf{r}'$. This allows us to rewrite Eq. (4.5) as follows:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \left[\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \mathbf{P}(\mathbf{r}')\cdot\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\right] d^3r' . \tag{4.7}$$

We can take this integral by parts. Taking into account

$$\nabla'\cdot\left\{\frac{\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right\} = \mathbf{P}(\mathbf{r}')\cdot\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) + \frac{1}{|\mathbf{r}-\mathbf{r}'|}\nabla'\cdot\mathbf{P}(\mathbf{r}') , \tag{4.8}$$

an integration by parts transforms the potential into

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \left\{\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{|\mathbf{r}-\mathbf{r}'|}\nabla'\cdot\mathbf{P}(\mathbf{r}') + \nabla'\cdot\left[\frac{\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right]\right\} d^3r' . \tag{4.9}$$

We can now use the divergence theorem to transform the third term in Eq. (4.9) to the integral over surface of the dielectric, which results in

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{d^3r'}{|\mathbf{r}-\mathbf{r}'|}\left[\rho(\mathbf{r}') - \nabla'\cdot\mathbf{P}(\mathbf{r}')\right] + \frac{1}{4\pi\varepsilon_0}\oint_S \frac{\mathbf{P}(\mathbf{r}')\cdot\mathbf{n}}{|\mathbf{r}-\mathbf{r}'|} da' . \tag{4.10}$$

As follows from this expression, the polarization of the medium produces an effective charge which can be interpreted as a macroscopic *bound* charge or *polarization* charge. There are two contributions to the bound charge – bulk and surface. The *volume charge density* is given by

$$\rho_P(\mathbf{r}) = -\nabla \cdot \mathbf{P}(\mathbf{r})\,. \tag{4.11}$$

The presence of the divergence of **P** in the effective charge density can be understood qualitatively. If the polarization is nonuniform there can be a net increase or decrease of charge within any small volume. For example, in Fig. 4.4a, at the center of this region, where the tails of the dipoles are concentrated, there is an excess of negative charge.

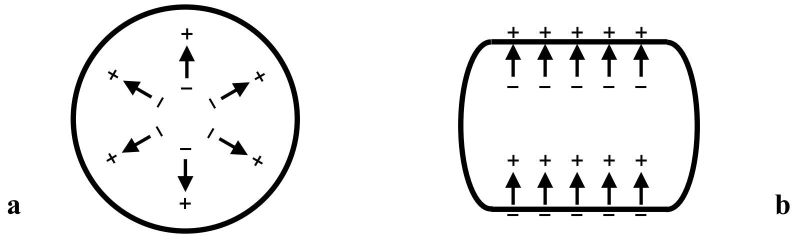

Fig. 4.4 Origin of polarization-charge density. (a) bulk charge density due to the divergence of polarization; (b) surface charge density due to uncompensated charges of the surface.

The *surface charge density* is

$$\sigma_P(\mathbf{r}) = \mathbf{P}(\mathbf{r}) \cdot \mathbf{n}\ . \tag{4.12}$$

This contribution is present even for the uniform polarization within a finite volume. In this case the average polarization charge inside the dielectric is zero, because if we take a macroscopic volume, it will contain equal amount of positive and negative charges and the net charge will be zero. On the other hand, if we consider a volume including a boundary perpendicular to the direction of polarization, there is a net positive (negative) charge on the surface which is not compensated by charges inside the dielectric, as is seen in Fig. 4b. Therefore, the polarization charge appears on the surface of the dielectric.

In deriving Eq. (4.10) we can integrate over all the space. In this case the surface integral (the third term in this equation) vanishes due to the assumption of the finite volume of the dielectric. The expression for the potential then becomes

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_{all\,space} \frac{d^3r'}{|\mathbf{r}-\mathbf{r}'|}\left[\rho(\mathbf{r}') - \nabla' \cdot \mathbf{P}(\mathbf{r}')\right]. \tag{4.13}$$

In this case, the surface polarization charge (4.12) is implicitly included in $-\nabla \cdot \mathbf{P}(\mathbf{r})$ due to the abrupt change of the polarization at the surface. This can be seen from the following consideration.

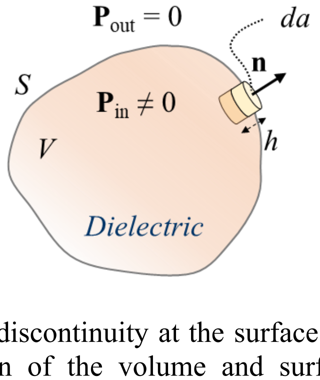

Fig. 4.5

Assume that polarization $\mathbf{P}(\mathbf{r})$ has discontinuity at the surface as is shown in Fig. 4.5. Consider a small pill box enclosing a small section of the volume and surface of the polarized material. From the divergence theorem we have

$$\int_V \nabla\cdot\mathbf{P}(\mathbf{r})d^3r = \int_S \mathbf{P}(\mathbf{r})\cdot\mathbf{n}da\ . \tag{4.14}$$

If the region is small enough this results in

$$\left(\mathbf{P}_{out}\cdot\mathbf{n}-\mathbf{P}_{in}\cdot\mathbf{n}\right)da = \nabla\cdot\mathbf{P}d^3r\ . \tag{4.15}$$

Taking into account that $\mathbf{P}_{out}=0$ and $\mathbf{P}_{in}=\mathbf{P}$ we have

$$\mathbf{P}\cdot\mathbf{n}da = -\nabla\cdot\mathbf{P}d^3r\ , \tag{4.16}$$

Thus $-\nabla\cdot\mathbf{P}$ must have a delta-function at the surface, and we still have the surface polarization charge so that on the surface

$$\sigma_P da = \rho_P d^3r\ . \tag{4.17}$$

Thus, the polarization charge can always be represented by

$$\rho_P = -\nabla\cdot\mathbf{P}\ . \tag{4.18}$$

It is important to note that the total polarization charge is always equal to zero. This is the consequence of charge conservation – by inducing an electric polarization in a material we do not change the total charge. Mathematically this fact can be easily seen from Eq. (4.18) – the integration of the polarization charge over any closed surface which enclosed the volume of a polarized material gives zero according to the divergence theorem.

We can, therefore, make a general statement that the presence of the polarization produces an additional polarization charge so that the total charge density becomes

$$\rho_{total} = \rho_{free} + \rho_P = \rho - \nabla\cdot\mathbf{P}\ . \tag{4.19}$$

We can, therefore, rewrite the expression for the divergence of **E** as follows:

$$\nabla\cdot\mathbf{E} = \frac{1}{\varepsilon_0}\left[\rho - \nabla\cdot\mathbf{P}\right]. \tag{4.20}$$

It is convenient to define the *electric displacement* **D**,

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}\ , \tag{4.21}$$

because this field is generated by the free charges only. Using the electric displacement, Gauss's law takes the form

$$\nabla\cdot\mathbf{D} = \rho\ . \tag{4.22}$$

In the integral form it reads as follows:

$$\oint_S \mathbf{D}\cdot\mathbf{n}da = \int_V \rho(\mathbf{r})d^3r\ . \tag{4.23}$$

This is particularly useful way to represent Gauss's law because it makes reference *only on free charges*.

Connecting **D** and **E** is necessary before a solution for the electrostatic potential or fields can be obtained. For a linear response of the system (4.3) the displacement **D** is proportional to **E**,

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} = \varepsilon_0\mathbf{E} + \varepsilon_0\chi_e\mathbf{E} = \varepsilon\mathbf{E}\ , \tag{4.24}$$

where

$$\varepsilon = \varepsilon_0(1+\chi_e) \tag{4.25}$$

is the electric permittivity. $\varepsilon_r = \varepsilon/\varepsilon_0 = 1 + \chi_e$ is called the *dielectric constant* or *relative* electric permittivity.

If the dielectric is not only isotropic, but also uniform, then ε is independent of position. Gauss's law (4.22) can then be written as

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon}. \tag{4.26}$$

In this case all problems *in that medium* are reduced to those with no electric polarization, except that the electric fields produced by given charges are reduced by a factor ε_r. The reduction can be understood in terms of a polarization of the atoms that produce fields in opposition to that of the given charge. One immediate consequence is that the capacitance of a capacitor is increased by a factor of ε_r if the empty space between the electrodes is filled with a dielectric with the dielectric constant ε_r.

Now we consider a couple of simple examples.

Example 1*. A slab with a uniform polarization pointing perpendicular to the surface*

Assume that we have a slab of a dielectric which has a uniform polarization pointing in the z direction (Fig. 4.6). In this case we have non-zero surface polarization charges, $\sigma_P = +P$ on the top surface and $\sigma_P = -P$ on the bottom surface, resulting in the electric field $\mathbf{E} = -\mathbf{P}/\varepsilon_0$ in the slab and $\mathbf{E} = 0$ outside the slab. The electric displacement $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} = 0$ is zero everywhere.

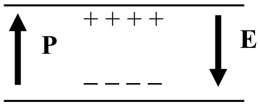

Fig. 4.6

If the polarization is uniform and parallel to the surfaces then the electric field **E** is zero, everywhere is space and thus $\mathbf{D} = \mathbf{P}$ inside the slab and $\mathbf{D} = 0$ outside.

Example 2*. Capacitor with a dielectric material inside*

Another simple example is the parallel plate capacitor with a linear dielectric of electric permittivity ε between the plates (Fig. 4.7). Assume that a battery voltage is applied to the plates of the capacitor so that the plates acquire a surface charge σ. According to Gauss's law, $\mathbf{D} = -\sigma\hat{\mathbf{z}}$, resulting in $\mathbf{E} = -\dfrac{\sigma}{\varepsilon}\hat{\mathbf{z}}$ and $\mathbf{P} = \mathbf{D} - \varepsilon_0\mathbf{E} = -\sigma\left(1 - \dfrac{\varepsilon_0}{\varepsilon}\right)\hat{\mathbf{z}}$. This leads to a polarization charge density on the surfaces of the dielectric:

$$\sigma_P = \mathbf{P} \cdot \mathbf{n} = \mp\sigma\left(1 - \frac{\varepsilon_0}{\varepsilon}\right). \tag{4.27}$$

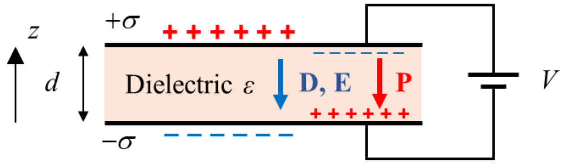

Fig. 4.7

As a result, the total charge (free and polarization charge) on the surfaces of the capacitor is reduced from σ in the absence of the dielectric to $\sigma + \sigma_P = \dfrac{\sigma}{\varepsilon_r}$ in its presence. Since electric field E, and hence voltage $V = Ed$, are determined by the total charge, the amount of free charge to maintain this voltage is reduced in the presence of dielectric, which implies that the capacitance is enhanced by the dielectric constant:

$$C = \frac{Q}{V} = \frac{\sigma A}{Ed} = \varepsilon\frac{A}{d}. \tag{4.28}$$

Example 3. *A sphere with a radial polarization distribution*

Another simple example is a sphere with the radial distribution of polarization (Fig. 4.8). The magnitude of polarization is constant and only its direction changes. Thus, the polarization is

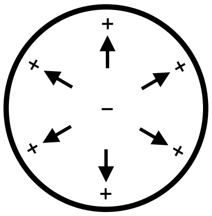

Fig. 4.8

$$\mathbf{P} = \begin{cases} 0, & outside; \\ P\hat{\mathbf{r}}, & inside. \end{cases} \tag{4.29}$$

This polarization of obviously produces a surface polarization charge on the surface of the sphere which is equal to $\sigma_P = \mathbf{P}\cdot\mathbf{n} = P$. The volume polarization charge can be found using Eq. (4.18) which leads to

$$\rho_P = -\nabla\cdot\mathbf{P} = -P\nabla\cdot\hat{\mathbf{r}} = -P\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\right) = -\frac{2P}{r}. \tag{4.30}$$

We see that the volume polarization charge is distributed over all the sphere and diverges at the center of the sphere. The total polarization charge is zero. Indeed, the total volume polarization charge is

$$\int \rho_P d^3r = \int_0^R \left(-\frac{2P}{r}\right) 4\pi r^2 dr = -4\pi R^2 P, \tag{4.31}$$

the total surface polarization charge is $4\pi R^2 P$.

Example 4. *Polarization by a point charge*

Consider a positive point charge *q* placed at the origin of an infinite dielectric medium of electric permittivity *ε*. We need to find polarization and polarization charges. Since the system has a spherical symmetry, we can use Gauss's law (4.22) to find the electric displacement **D**:

$$\oint_S \mathbf{D}\cdot\mathbf{n}da = q, \tag{4.32}$$

where the integration is performed over a sphere of radius *r* centered at the origin. This leads to

$$\mathbf{D} = \frac{q}{4\pi r^2}\hat{\mathbf{r}}. \tag{4.33}$$

Consequently, the electric field is

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon} = \frac{q}{4\pi\varepsilon r^2}\hat{\mathbf{r}}. \tag{4.34}$$

According to Eq. (4.21), the polarization is

$$\mathbf{P} = \mathbf{D} - \varepsilon_0\mathbf{E} = \frac{q}{4\pi r^2}\hat{\mathbf{r}} - \frac{\varepsilon_0 q}{4\pi\varepsilon r^2}\hat{\mathbf{r}} = \frac{q}{4\pi}\left(1 - \frac{\varepsilon_0}{\varepsilon}\right)\frac{\hat{\mathbf{r}}}{r^2} = \frac{q}{4\pi}\left(\frac{\varepsilon - \varepsilon_0}{\varepsilon}\right)\frac{\hat{\mathbf{r}}}{r^2}. \tag{4.35}$$

The polarization charge is

$$\rho_P = -\nabla\cdot\mathbf{P} = -\frac{q}{4\pi}\left(\frac{\varepsilon - \varepsilon_0}{\varepsilon}\right)\nabla\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right) = -q\left(\frac{\varepsilon - \varepsilon_0}{\varepsilon}\right)\delta^3(\mathbf{r}). \tag{4.36}$$

We have therefore a polarization at the origin which has an opposite sign to the free charge *q*. The sum of the free and polarization charge is $\frac{\varepsilon_0}{\varepsilon}q = \frac{q}{\varepsilon_r}$, which gives a correct electric field (4.34) if we use this charge in Coulomb's law. We note that the total polarization charge is not zero because the system considered is infinite and hence the positive polarization charge is located at infinity.

Boundary Conditions

For solving electrostatics problems, one needs to know boundary conditions for the electric field.

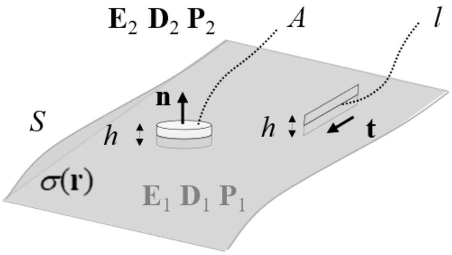

Fig. 4.9 Schematic diagram of the boundary surface between different media.

Consider a boundary between different media, as is shown in Fig. 4.9. The boundary region is assumed to carry idealized surface charge σ. Consider a small pillbox, half in one medium and half in the other, with the normal it to its top pointing from medium 1 into medium 2. According to the Gauss's law

$$\oint_{S'} \mathbf{D}\cdot\mathbf{n}' da' = \sigma A\,, \tag{4.37}$$

where the integral is taken over the surface of the pillbox S' and A is the area of the pillbox lid. In the limit of zero thickness, the sides of the pillbox contribute nothing to the flux. The contribution from the top and bottom surfaces to the integral gives $A\left(\mathbf{D}_2 - \mathbf{D}_1\right)\cdot\mathbf{n}$, where **n** is the normal to the surface S, resulting in

$$D_2^{\perp} - D_1^{\perp} = \sigma\,, \tag{4.38}$$

where $D^{\perp}$ is the component of the electrical displacement perpendicular to the surface. Eq. (4.38) tells us that there is a discontinuity of the $D^{\perp}$ at the interface which is determined by the surface charge.

Now we consider a rectangular contour C such that it is partly in one medium and partly in the other and is oriented with its plane perpendicular to the surface. Since the curl of electric field is zero, we have

$$\oint \mathbf{E}\cdot d\mathbf{l} = 0\,. \tag{4.39}$$

For the rectangular contour C of infinitesimal height this integral is equal to $\left(E_2^{\parallel} - E_1^{\parallel}\right)l$, where $E^{\parallel}$ is the component of electric field parallel to the surface. This implies that

$$\mathbf{E}_2^{\parallel} = \mathbf{E}_1^{\parallel}\,, \tag{4.40}$$

i.e. the tangential component of electric field is always continuous.

The electrostatic potential is continuous across the boundary. Indeed, if we consider two points, point A above the surface and point B below the surface, then

$$\Phi(B) - \Phi(A) = -\int_A^B \mathbf{E}\cdot d\mathbf{l}\,. \tag{4.41}$$

As the path length shrinks to zero, the integral does too.

Boundary-Value Problems with Dielectrics

Since the presence of dielectrics equations for the electric field are $\nabla \times \mathbf{E} = 0$ and $\nabla \cdot \mathbf{E} = \rho / \varepsilon$, the electrostatic potential defined by $\mathbf{E} = -\nabla \Phi$ obeys the Poisson equation

$$\nabla^2 \Phi = -\rho / \varepsilon . \tag{4.42}$$

Therefore, all the methods of solving boundary value problems in electrostatics discussed in preceding sections can readily be extended to dielectric materials. Below we consider a few examples.

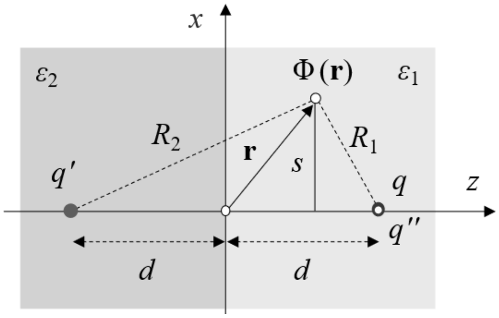

Fig. 4.10

***Example 1*:** To illustrate the method of images for dielectrics we consider a point charge q embedded in a semi-infinite dielectric, with dielectric constant ε_1, a distance d away from a plane interface that separates the first medium from another semi-infinite dielectric ε_2. The surface may be taken as the plane $z = 0$, as shown in Fig. 4.10. We must find the appropriate solution to the equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_1}, \quad z > 0, \tag{4.43}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_2}, \quad z < 0, \tag{4.44}$$

and

$$\nabla \times \mathbf{E} = 0 . \tag{4.45}$$

They should satisfy the boundary conditions (4.38) and (4.40) at $z = 0$:

$$D_2^{\perp} = \varepsilon_2 E_{2z} = D_1^{\perp} = \varepsilon_1 E_{1z}, \tag{4.46}$$

$$\begin{aligned} E_{2x} &= E_{1x}, \\ E_{2y} &= E_{1y}, \end{aligned} \tag{4.47}$$

where we took into account that $\sigma = 0$. $\mathbf{E}$ can be obtained from the potential $\mathbf{E} = -\nabla \Phi$. We attempt to use the method of images by locating an image charge q' at the symmetrical position A' shown in Fig. 4.10. Then for $z > 0$ the potential at a point P described by cylindrical coordinates (s, ϕ, z) will be

$$\Phi = \frac{1}{4\pi\varepsilon_1}\left(\frac{q}{R_1} + \frac{q'}{R_2}\right), \quad z > 0 . \tag{4.48}$$

where $R_1 = \sqrt{s^2 + (d - z)^2}$ and $R_2 = \sqrt{s^2 + (d + z)^2}$. So far, the procedure is completely analogous to the problem with a conducting material in place of the dielectric ε_2 for $z < 0$. But we now must specify the potential for $z < 0$. Since there are no charges in the region $z < 0$, it must be a solution of the Laplace equation without singularities in that region. Clearly the simplest assumption is that for $z < 0$ the potential is equivalent to that of a charge q'' at the position A of the actual charge q:

$$\Phi = \frac{1}{4\pi\varepsilon_2} \frac{q''}{R_1}, \quad z < 0 . \tag{4.49}$$

Now we try to pick the image charges in such a way that the boundary conditions (4.46) and (4.47) are satisfied. These conditions involve the following derivatives:

$$\frac{\partial}{\partial z}\left(\frac{1}{R_1}\right)_{z=0} = -\frac{\partial}{\partial z}\left(\frac{1}{R_2}\right)_{z=0} = \frac{d}{\left(s^2+d^2\right)^{3/2}} \tag{4.50}$$

and

$$\frac{\partial}{\partial s}\left(\frac{1}{R_1}\right)_{z=0} = \frac{\partial}{\partial s}\left(\frac{1}{R_2}\right)_{z=0} = \frac{-s}{\left(s^2+d^2\right)^{3/2}}\,. \tag{4.51}$$

Using these equations, we find that the condition for the continuous normal component of **D** (4.46), which is determined by the z-component of electric field $E_z = -\partial\Phi/\partial z$, leads to

$$q - q' = q''\,, \tag{4.52}$$

and the condition for the continuous tangential component of **E** (4.47), given by $E_s = -\frac{\partial\Phi}{\partial s}$, results in

$$\frac{\left(q+q'\right)}{\varepsilon_1} = \frac{q''}{\varepsilon_2}\,. \tag{4.53}$$

Note that equivalently we could have used the condition of the continuity of the potential which would lead to the same Eq. (4.53). These equations can be solved to yield the image charges q' and q'':

$$\begin{aligned} q' &= \frac{\left(\varepsilon_1-\varepsilon_2\right)}{\left(\varepsilon_1+\varepsilon_2\right)}q\,, \\ q'' &= \frac{2\varepsilon_2}{\left(\varepsilon_1+\varepsilon_2\right)}q\,. \end{aligned} \tag{4.54}$$

We can now obtain the potential which is given by

$$\Phi_1(z) = \frac{1}{4\pi\varepsilon_1}\left(\frac{q}{\sqrt{s^2+(d-z)^2}} + \frac{\left(\varepsilon_1-\varepsilon_2\right)}{\left(\varepsilon_1+\varepsilon_2\right)}\frac{q}{\sqrt{s^2+(d+z)^2}}\right), \quad z>0\,; \tag{4.55}$$

$$\Phi_2(z) = \frac{1}{4\pi\varepsilon_2}\frac{2\varepsilon_2}{\left(\varepsilon_1+\varepsilon_2\right)}\frac{q}{\sqrt{s^2+(d-z)^2}}, \quad z<0\,. \tag{4.56}$$

For the two cases $\varepsilon_2 > \varepsilon_1$ and $\varepsilon_2 < \varepsilon_1$ the field lines of **D** are shown schematically in Fig. 4.11.

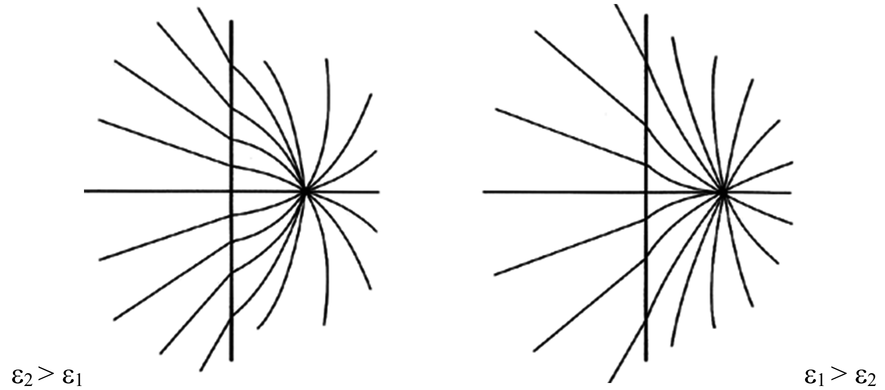

Fig. 4.11. Lines of **D** for a point charge embedded in a dielectric 1 near a boundary with dielectric 2.

The polarization-charge density is given by $\rho_P = -\nabla\cdot\mathbf{P} = -\nabla\cdot\left[(1-\varepsilon_0/\varepsilon)\mathbf{D}\right]$. In the regions where ε is constant it can be taken out of differentiation which leads to $\rho_P = -(1-\varepsilon_0/\varepsilon)\nabla\cdot\mathbf{D} = 0$, where we took into account the Gauss law. Thus, for a linear and uniform dielectric, in the absence of volume free charges, it must be no volume polarization charge. In our problem the polarization charge will be only at the point of charge q. At the interface between the two dielectrics, however, ε takes a discontinuous jump, as z passes through $z = 0$. This implies that there is a polarization-surface-charge density on the plane $z = 0$. The latter can be calculated by generalizing Eq. (4.12) to the case of two dielectrics:

$$\sigma_P = \left(\mathbf{P}_1 - \mathbf{P}_2\right)\cdot\mathbf{n}. \tag{4.57}$$

where $\mathbf{n} = -\hat{\mathbf{z}}$ is the unit normal pointing from dielectric 1 to dielectric 2, and $\mathbf{P}_i$ (i =1,2) is the polarization in the dielectric i at $z = 0$. Since

$$\mathbf{P}_i = \left(\varepsilon_i - \varepsilon_0\right)\mathbf{E}_i = -\left(\varepsilon_i - \varepsilon_0\right)\nabla\cdot\Phi_i\big|_{z=0}, \tag{4.58}$$

we find from Eqs. (4.55) and (4.56) that

$$\begin{aligned} P_{1z} &= -\left(\varepsilon_1 - \varepsilon_0\right)\frac{d\Phi_1}{dz}\bigg|_{z=0} = -\left(\varepsilon_1 - \varepsilon_0\right)\frac{q}{4\pi\varepsilon_1}\left\{\frac{(d-z)}{\left[s^2+(d-z)^2\right]^{3/2}} - \frac{\left(\varepsilon_1-\varepsilon_2\right)}{\left(\varepsilon_1+\varepsilon_2\right)}\frac{(d+z)}{\left[s^2+(d+z)^2\right]^{3/2}}\right\}_{z=0} = \\ &= -\frac{q}{2\pi}\frac{\varepsilon_2\left(\varepsilon_1-\varepsilon_0\right)}{\varepsilon_1\left(\varepsilon_1+\varepsilon_2\right)}\frac{d}{\left[s^2+d^2\right]^{3/2}}, \end{aligned} \tag{4.59}$$

$$P_{2z} = -\left(\varepsilon_2 - \varepsilon_0\right)\frac{d\Phi_2}{dz}\bigg|_{z=0} = -\left(\varepsilon_2 - \varepsilon_0\right)\frac{q}{4\pi\varepsilon_2}\frac{2\varepsilon_2}{\left(\varepsilon_1+\varepsilon_2\right)}\frac{(d-z)}{\left[s^2+(d-z)^2\right]^{3/2}}\Bigg|_{z=0} = -\frac{q}{2\pi}\frac{\left(\varepsilon_2-\varepsilon_0\right)}{\left(\varepsilon_1+\varepsilon_2\right)}\frac{d}{\left[s^2+d^2\right]^{3/2}}. \tag{4.60}$$

Therefore,

$$\sigma_P = -P_{1z} + P_{2z} = \frac{q}{2\pi}\frac{\varepsilon_0\left(\varepsilon_1-\varepsilon_2\right)}{\varepsilon_1\left(\varepsilon_1+\varepsilon_2\right)}\frac{d}{\left[s^2+d^2\right]^{3/2}}. \tag{4.61}$$

In the limit $\varepsilon_1 = \varepsilon_0$ and $\varepsilon_2 \to \infty$ the electric field inside dielectric 2 becomes very small and hence it behaves much like a conductor. Then the surface-charge density (4.61) approaches the value appropriate to a conducting surface,

$$\sigma_P = -\frac{q}{2\pi}\frac{d}{\left[s^2+d^2\right]^{3/2}}. \tag{4.62}$$

Example 2. The second illustration of electrostatic problems involving dielectrics is that of a dielectric sphere of radius a with dielectric constant $\varepsilon_r = \varepsilon/\varepsilon_0$ placed in a uniform electric field, which at large distances from the sphere is directed along the z axis and has magnitude E_0, as indicated in Fig. 4.12.

Both inside and outside the sphere there are no free charges. Consequently, the problem is one of solving the Laplace equation with the proper boundary conditions at $r = a$. From the axial symmetry of the geometry, we can take the solution to be of the form a Legendre polynomial expansion. The solution is different for the region inside and outside the sphere and take the form:

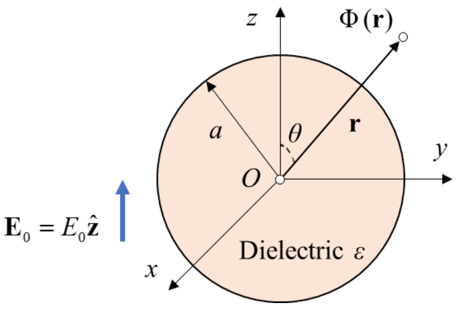

Fig. 4.12

$$\Phi_{in}(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)\,,\ \ r < a \tag{4.63}$$

$$\Phi_{out}(r,\theta) = \sum_{l=0}^{\infty} \left(B_l r^l + C_l r^{-l-1} \right) P_l(\cos\theta)\,,\ \ r > a \tag{4.64}$$

The coefficients A_l, B_l, and C_l are to be determined from the boundary conditions. From the boundary condition at infinity suggesting that

$$\Phi \to -E_0 r\cos\theta,\ \ r \gg a \tag{4.65}$$

we find that the only non-vanishing B_l is $B_1 = -E_0$. The other coefficients are determined from the boundary conditions at $r = a$. The electrostatic potential should be continuous on the surface and hence

$$\Phi_{in}\big|_{r=a} = \Phi_{out}\big|_{r=a}\,. \tag{4.66}$$

Since there are no free surface charges, the normal component of the electric displacement should also be continuous on the surface and hence

$$-\varepsilon \frac{\partial \Phi_{in}}{\partial r}\bigg|_{r=a} = -\varepsilon_0 \frac{\partial \Phi_{out}}{\partial r}\bigg|_{r=a}\,. \tag{4.67}$$

Now we substitute the series (4.63) and (4.64) in these boundary conditions. For the first boundary condition (4.63) this leads to

$$\sum_{l=0}^{\infty} A_l a^l P_l(\cos\theta) = \sum_{l=0}^{\infty} \left(-E_0 a^l \delta_{1l} + C_l \frac{1}{a^{l+1}} \right) P_l(\cos\theta)\,. \tag{4.68}$$

Since this equation must be valid for all θ, the coefficients must be equal at each l resulting in

$$\begin{aligned} A_1 &= -E_0 + \frac{C_1}{a^3}, \\ A_l &= \frac{C_l}{a^{2l+1}},\ \ l \neq 1. \end{aligned} \tag{4.69}$$

The second boundary condition (4.64) leads to

$$\varepsilon \sum_{l=0}^{\infty} l A_l a^{l-1} P_l(\cos\theta) = \varepsilon_0 \sum_{l=0}^{\infty} \left(-E_0 \delta_{1l} - (l+1) C_l \frac{1}{a^{l+2}} \right) P_l(\cos\theta)\,. \tag{4.70}$$

Since this equation must be valid for all θ, the coefficients must be equal at each l resulting in

$$\begin{aligned} \frac{\varepsilon}{\varepsilon_0} A_1 &= -E_0 - \frac{2C_1}{a^3}, \\ \frac{\varepsilon}{\varepsilon_0} l A_l &= -(l+1) \frac{C_l}{a^{2l+1}},\ \ l \neq 1. \end{aligned} \tag{4.71}$$

The second equations in (4.69) and (4.71) can be satisfied simultaneously only with $A_l = C_l = 0$ for all $l \neq 1$. The remaining coefficients are given in terms of the applied electric field E_0

$$\begin{aligned} A_1 &= -\frac{3}{\varepsilon_r + 2} E_0\,, \\ C_1 &= \frac{\varepsilon_r - 1}{\varepsilon_r + 2} a^3 E_0\,. \end{aligned} \tag{4.72}$$

The potential is therefore

$$\Phi_{in}(r,\theta) = -\frac{3}{\varepsilon_r + 2} E_0 r\cos\theta\,,\quad r < a\,, \tag{4.73}$$

$$\Phi_{out}(r,\theta) = -E_0 r\cos\theta + \frac{\varepsilon_r - 1}{\varepsilon_r + 2} E_0 \frac{a^3}{r^2}\cos\theta\,,\quad r > a\,. \tag{4.74}$$

It is easy to see that the potential inside the sphere describes a constant electric field parallel to the applied field with magnitude

$$\mathbf{E}_{in} = \frac{3}{\varepsilon_r + 2}\mathbf{E}_0\,. \tag{4.75}$$

Outside the sphere the potential is equivalent to the applied field $\mathbf{E}_0$ plus the field of an electric dipole at the origin with dipole moment:

$$\mathbf{p} = 4\pi\varepsilon_0 \frac{\varepsilon_r - 1}{\varepsilon_r + 2} a^3 \mathbf{E}_0\,, \tag{4.76}$$

oriented in the direction of the applied field.

The dipole moment can be interpreted as the volume integral of the polarization **P**. The polarization is

$$\mathbf{P} = (\varepsilon - \varepsilon_0)\mathbf{E} = 3\varepsilon_0 \frac{(\varepsilon_r - 1)}{\varepsilon_r + 2}\mathbf{E}_0\,. \tag{4.77}$$

It is constant throughout the volume of the sphere and has a volume integral given by Eq. (4.76). The polarization-surface-charge density is

$$\sigma_P = \mathbf{P}\cdot\mathbf{n} = 3\varepsilon_0 \frac{(\varepsilon_r - 1)}{\varepsilon_r + 2} E_0 \cos\theta\,. \tag{4.78}$$

This can be thought of as producing an internal field (known as *depolarizing* field) directed oppositely to the applied field, so reducing the field inside the sphere to its value (4.75), as sketched in Fig. 4.13a.

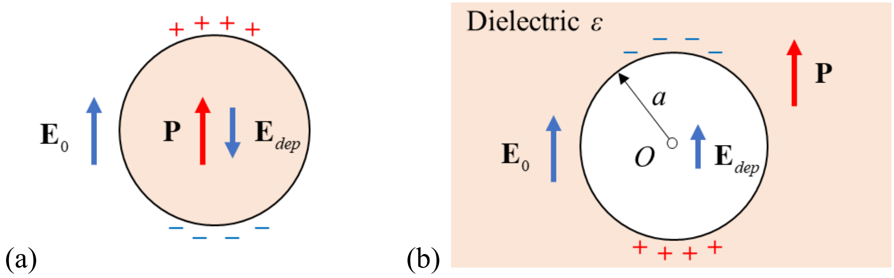

Fig. 4.13 (a) Dielectric sphere in a uniform field E_0, showing polarization and polarization charge producing a depolarizing field. (b) Spherical cavity in a dielectric with a uniform field applied.

The problem of a spherical cavity of radius a in a dielectric medium with dielectric constant $\varepsilon_r = \varepsilon/\varepsilon_0$ and with an applied electric field $\mathbf{E}_0$ parallel to the z axis, as shown in Fig. 4.13b, can be handled in the same way as the dielectric sphere. In fact, inspection of boundary conditions (4.66) and (4.67) shows that the results for the cavity can be obtained from those of the sphere by the replacement $\varepsilon/\varepsilon_0$ to $\varepsilon_0/\varepsilon$. Thus, for example, the field inside the cavity is uniform, parallel to $\mathbf{E}_0$, and of magnitude:

$$\mathbf{E}_{in} = \frac{3\varepsilon_r}{2\varepsilon_r + 1}\mathbf{E}_0\,. \tag{4.79}$$

Similarly, the field outside is the applied field plus that of a dipole at the origin *oriented oppositely* to the applied field and with dipole moment

$$\mathbf{p} = -4\pi\varepsilon_0 \frac{\varepsilon_r - 1}{2\varepsilon_r + 1} a^3 \mathbf{E}_0 \, . \tag{4.80}$$

Electrostatic Energy in Dielectrics

In free space we derived the energy of a distribution of charge $\rho(\mathbf{r})$ by assembling the distribution little by little, bringing infinitesimal pieces of charge in from infinity. Following this reasoning we found that

$$U = \frac{1}{2}\int \rho(\mathbf{r})\Phi(\mathbf{r}) d^3 r \, . \tag{4.81}$$

This is in general not true in the presence of dielectrics (however, as we will see, it may be true in some cases). In the presence of dielectrics work must also be done to induce polarization in the dielectric, and it is not clear if this work is included in the equation above.

When dielectrics are present, we will use a somewhat different argument (which still corresponds to the same procedure). Suppose that there is initially some macroscopic charge density $\rho(\mathbf{r})$, potential $\Phi(\mathbf{r})$, and fields $\mathbf{E}(\mathbf{r})$ and $\mathbf{D}(\mathbf{r})$. Imagine that some infinitesimal change in the charge density, $\delta\rho(\mathbf{r})$, is made. To first order in $\delta\rho(\mathbf{r})$, the change in energy of the system is

$$\delta U = \int \delta\rho(\mathbf{r})\Phi(\mathbf{r}) d^3 r \, , \tag{4.82}$$

where the integration is performed over all space. The point is that this is the interaction energy of $\delta\rho(\mathbf{r})$ with the sources already present (and which produce $\Phi(\mathbf{r})$); the interaction energy of $\delta\rho(\mathbf{r})$ with itself is second-order in small (infinitesimal) quantities.

The change in **D** which arises as a consequence of the change $\delta\rho(\mathbf{r})$ in the charge density is related to the latter by the equation $\nabla\cdot\left(\mathbf{D}+\delta\mathbf{D}\right) = \rho + \delta\rho$ and therefore $\delta\rho = \nabla\cdot\delta\mathbf{D}$, where $\delta\mathbf{D}$ is the resulting change in **D**, so we can write the change in the energy as

$$\delta U = \int \left(\nabla\cdot\delta\mathbf{D}\right)\Phi d^3 r \, . \tag{4.83}$$

Now

$$\nabla\cdot\left(\delta\mathbf{D}\Phi\right) = \left(\nabla\cdot\delta\mathbf{D}\right)\Phi + \delta\mathbf{D}\cdot\left(\nabla\Phi\right) = \left(\nabla\cdot\delta\mathbf{D}\right)\Phi - \delta\mathbf{D}\cdot\mathbf{E} \, , \tag{4.84}$$

and hence integrating by parts we obtain

$$\delta U = \int \nabla\cdot\left(\delta\mathbf{D}\Phi\right) d^3 r + \int \delta\mathbf{D}\cdot\mathbf{E} d^3 r \, . \tag{4.85}$$

The divergence theorem turns the first term into a surface integral, which vanishes for a localized charge distribution if the integration is performed over all space:

$$\int_V \nabla\cdot\left(\delta\mathbf{D}\Phi\right) d^3 r = \oint_S \left(\delta\mathbf{D}\Phi\right)\cdot\mathbf{n} da = 0 \, . \tag{4.86}$$

Therefore, the work done is equal to

$$\delta U = \int \delta\mathbf{D}\cdot\mathbf{E} d^3 r \, . \tag{4.87}$$

So far, it applies to *any* material. Now, if the material is a linear dielectric, then $\mathbf{D} = \varepsilon\mathbf{E}$, so infinitesimal increments are

$$\frac{1}{2}\delta\left(\mathbf{D}\cdot\mathbf{E}\right) = \frac{1}{2}\delta\left(\varepsilon E^2\right) = \varepsilon\delta\mathbf{E}\cdot\mathbf{E} = \delta\mathbf{D}\cdot\mathbf{E} \, . \tag{4.88}$$

Thus, the change in the electrostatic energy is

$$\delta U = \frac{1}{2}\int \delta\left(\mathbf{D}\cdot\mathbf{E}\right) d^3r \,. \tag{4.89}$$

Now we build the free charge up from zero to the final configuration. This corresponds to integrating from zero field up to the final field (a functional integration),

$$U = \frac{1}{2}\int d^3r \int_0^{\mathbf{D}} \delta\left(\mathbf{D}\cdot\mathbf{E}\right). \tag{4.90}$$

This leads to

$$U = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E} d^3r \,. \tag{4.91}$$

This result is valid only for linear media and results in Eq. (4.81). Indeed, using $\mathbf{E} = -\nabla\Phi$ and integrating Eq. (4.91) by parts, we obtain

$$U = -\frac{1}{2}\int \mathbf{D}\cdot\nabla\Phi d^3r = -\frac{1}{2}\int \nabla\cdot\left(\mathbf{D}\Phi\right) d^3r + \frac{1}{2}\int \Phi\nabla\cdot\mathbf{D} d^3r \,. \tag{4.92}$$

Through the divergence theorem, the first term yields a surface term which vanishes at infinity. The second term becomes

$$U = \frac{1}{2}\int \Phi\nabla\cdot\mathbf{D} d^3r = \frac{1}{2}\int \rho(\mathbf{r})\Phi(\mathbf{r}) d^3r \,. \tag{4.93}$$

Thus, for a linear dielectric, the original formula is valid.