Section 4: Electromagnetic Waves in Conductors and Dispersive Matter

4.1 Electromagnetic waves in conductors

In our discussion of electromagnetic waves in vacuum or in dielectric materials, we assumed that the free charge density ρ_f and the free current density $\mathbf{J}_f$ are zero. This assumption is not valid for conductors where the free currents can affect the propagation of electromagnetic waves. In general, current flow in neutral matter is a complex phenomenon whose proper microscopic description requires arguments from statistical physics and quantum mechanics. Nevertheless, it is well established phenomenologically that the current density in many systems obeys Ohm's law. According to Ohm's law, the (free) current density in a conductor is proportional to the electric field:

$$\mathbf{J}_f = \sigma \mathbf{E}, \tag{4.1}$$

where σ is the electric conductivity of the conductor (do not mix up with the surface charge density which is denoted by the same Greek letter). In general, σ is a second rank tensor, but for *isotropic* conductors, it can be regarded as a scalar. With this, Maxwell's equations for linear media have the form

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon}\rho_f , \tag{4.2}$$

$$\nabla \cdot \mathbf{B} = 0 , \tag{4.3}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{4.4}$$

$$\nabla \times \mathbf{B} = \mu\sigma\mathbf{E} + \varepsilon\mu\frac{\partial \mathbf{E}}{\partial t} . \tag{4.5}$$

Now, the continuity equation for free charge,

$$\nabla \cdot \mathbf{J}_f = -\frac{\partial \rho_f}{\partial t}, \tag{4.6}$$

together with Ohm's law (4.1) and Gauss's law (4.2), gives

$$\frac{\partial \rho_f}{\partial t} = -\sigma\nabla \cdot \mathbf{E} = -\frac{\sigma}{\varepsilon}\rho_f , \tag{4.7}$$

where we assumed the conductor be *linear* and *homogeneous*, so that σ and ε are constant in space. It follows from Eq. (4.7) that

$$\rho_f(t) = e^{-(\sigma/\varepsilon)t}\rho_f(0) . \tag{4.8}$$

Thus, any initial free charge $\rho_f(0)$ dissipates in a characteristic damping time $\tau_d = \varepsilon/\sigma$. This reflects the familiar fact that if we put some free charge on a conductor, it will flow out to the edges. The time constant τ_d is a measure of how "good" a conductor is: For a "perfect" conductor, $\sigma = \infty$ and $\tau_d = 0$; for a "good" conductor, τ_d is much less than the other relevant times in the problem. Since we are not interested in this transient behavior—we will wait for any accumulated free charge to disappear. Therefore, we assume $\rho_f = 0$, so that Maxwell's equation are

$$\nabla \cdot \mathbf{E} = 0 , \tag{4.9}$$

$$\nabla \cdot \mathbf{B} = 0 , \tag{4.10}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{4.11}$$

$$\nabla \times \mathbf{B} = \mu\sigma\mathbf{E} + \varepsilon\mu\frac{\partial \mathbf{E}}{\partial t} . \tag{4.12}$$

These differ from the corresponding equations for nonconducting media only by the $\mu\sigma\mathbf{E}$ term in (4.12), which is absent, obviously, when $\sigma = 0$.

Applying the curl to Eqs. (4.11) and (4.12), as before, we obtain:

$$\nabla\times(\nabla\times\mathbf{E}) = \nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E} = -\nabla\times\frac{\partial\mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla\times\mathbf{B}) = -\mu\sigma\frac{\partial\mathbf{E}}{\partial t} - \mu\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2}, \tag{4.13}$$

$$\nabla\times(\nabla\times\mathbf{B}) = \nabla(\nabla\cdot\mathbf{B}) - \nabla^2\mathbf{B} = \nabla\times(\mu\sigma\mathbf{E}) + \nabla\times\left(\mu\varepsilon\frac{\partial\mathbf{E}}{\partial t}\right) = \mu\sigma\nabla\times\mathbf{E} + \mu\varepsilon\frac{\partial}{\partial t}(\nabla\times\mathbf{E}) = -\mu\sigma\frac{\partial\mathbf{B}}{\partial t} - \mu\varepsilon\frac{\partial^2\mathbf{B}}{\partial t^2}. \tag{4.14}$$

This leads to modified wave equations for **E** and **B**:

$$\nabla^2\mathbf{E} = \mu\sigma\frac{\partial\mathbf{E}}{\partial t} + \mu\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2}, \tag{4.15}$$

$$\nabla^2\mathbf{B} = \mu\sigma\frac{\partial\mathbf{B}}{\partial t} + \mu\varepsilon\frac{\partial^2\mathbf{B}}{\partial t^2}. \tag{4.16}$$

These equations still admit plane-wave solutions:

$$\mathbf{E}(z,t) = \mathbf{E}_0 e^{i(kz-\omega t)}, \tag{4.17}$$

$$\mathbf{B}(z,t) = \mathbf{B}_0 e^{i(kz-\omega t)}, \tag{4.18}$$

where $\mathbf{E}_0$ and $\mathbf{B}_0$ are the complex amplitudes. Now, however, the "wave number" k is complex:

$$k^2 = \mu\varepsilon\omega^2 + i\mu\sigma\omega, \tag{4.19}$$

as can be easily checked by plugging Eqs. (4.17) and (4.18) into Eqs. (4.15) and (4.16), respectively. The real k' and imaginary k'' parts of the wave vector k,

$$k = k' + ik'', \tag{4.20}$$

can be found from Eq. (4.19), which gives

$$k'^2 - k''^2 = \mu\varepsilon\omega^2, \tag{4.21}$$

$$k'k'' = \frac{\mu\sigma\omega}{2}, \tag{4.22}$$

resulting in

$$k' = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1\right]^{1/2}, \tag{4.23}$$

$$k'' = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1\right]^{1/2}. \tag{4.24}$$

The real part of k determines the wavelength, the propagation speed, and the index of refraction:

$$\lambda = \frac{2\pi}{k'}, \quad v = \frac{\omega}{k'}, \quad n' = \frac{c}{\omega}k'. \tag{4.25}$$

The imaginary part of k results in an attenuation of the wave (decreasing amplitude with increasing z):

$$\mathbf{E}(z,t) = \mathbf{E}_0 e^{-k''z} e^{i(k'z-\omega t)}, \tag{4.26}$$

$$\mathbf{B}(z,t) = \mathbf{B}_0 e^{-k''z} e^{i(k'z-\omega t)}. \tag{4.27}$$

The distance it takes to reduce the amplitude by a factor of *e* is called the *skin depth*:

$$\delta \equiv \frac{1}{k''} . \tag{4.28}$$

It is a measure of how far the wave penetrates into the conductor: $E \propto E_0 e^{-z/\delta}$.

For a good conductor, such as cupper, typically we have $\frac{\sigma}{\varepsilon\omega} \gg 1$ up to optical light frequencies ω ~10^{15}. In this case, the skin depth is

$$\delta(\omega) \approx \sqrt{\frac{2}{\mu\sigma\omega}} . \tag{4.29}$$

In practice, δ has a small value (for Cu at radio frequency ω ~10^7 s^{-1}, δ ~ 100 μm, and at visible light frequency ω ~10^{15} s^{-1}, δ ~ 10 nm), indicating that an electromagnetic wave incident on a metallic specimen penetrates only a short distance below the surface.

The attenuated plane waves (4.26) and (4.27) satisfy the modified wave equations (4.15) and (4.16) for any $\mathbf{E}_0$ and $\mathbf{B}_0$. But Maxwell's equations (4.9)–(4.12) impose further constraints, which serve to determine the relative amplitudes, phases, and polarizations of **E** and **B**. As before, Eqs. (4.9) and (4.10) rule out any *z* components due to

$$\nabla\cdot\mathbf{E} = \left(\nabla\cdot\mathbf{E}_0\right)e^{i(kz-\omega t)} + \mathbf{E}_0\cdot\nabla e^{i(kz-\omega t)} = \mathbf{E}_0\cdot\hat{\mathbf{z}}e^{i(kz-\omega t)}ik = 0 . \tag{4.30}$$

Therefore, the fields are *transverse*. We may as well orient our axes so that **E** is polarized along the *x* direction:

$$\mathbf{E}(z,t) = E_0 e^{-k''z} e^{i(k'z-\omega t)}\hat{\mathbf{x}} . \tag{4.31}$$

Then Eq. (4.11) gives

$$\nabla\times\mathbf{E} = -\mathbf{E}_0\times\nabla e^{i(kz-\omega t)} = -ik\mathbf{E}_0\times\hat{\mathbf{z}}e^{i(kz-\omega t)} = i\omega\mathbf{B}_0 e^{i(kz-\omega t)} \tag{4.32}$$

or

$$\mathbf{B}_0 = \frac{k}{\omega}\left(\hat{\mathbf{z}}\times\mathbf{E}_0\right) . \tag{4.33}$$

Therefore,

$$\mathbf{B}(z,t) = \frac{k}{\omega}E_0 e^{-k''z} e^{i(k'z-\omega t)}\hat{\mathbf{y}} . \tag{4.34}$$

As before, the electric and magnetic fields are mutually perpendicular.

Like any complex number, *k* can be expressed in terms of its modulus and phase:

$$k = |k|e^{i\phi} , \tag{4.35}$$

where

$$|k| = \sqrt{k'^2 + k''^2} = \omega\sqrt{\varepsilon\mu\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}} , \tag{4.36}$$

and

$$\phi = \tan^{-1}\frac{k''}{k'} . \tag{4.37}$$

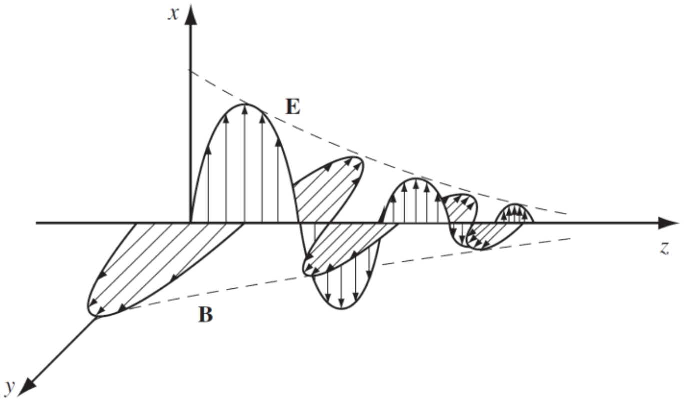

Fig. 4.1

Using (4.35), we obtain from Eq. (4.34) that

$$\mathbf{B}(z,t) = \frac{|k|e^{i\phi}}{\omega} E_0 e^{-k''z} e^{i(k'z-\omega t)} \hat{\mathbf{y}} . \tag{4.38}$$

Comparing to Eq.(4.31), see that the electric and magnetic fields are no longer in phase. The magnetic field *lags behind* the electric field, so that there is a phase difference ϕ between the two. For example, in good conductors, where $\frac{\sigma}{\varepsilon\omega} \gg 1$, as follows from Eqs. (4.23) and (4.24), the real and imaginary parts of the complex wave vector are

$$k' \approx k'' \approx \sqrt{\frac{\mu\sigma\omega}{2}} . \tag{4.39}$$

In this case, the phase difference between the electric and magnetic field is $\phi \approx \tan^{-1} 1 = 45°$.

The (real) electric and magnetic fields are

$$\mathbf{E}(z,t) = E_0 e^{-k''z} \cos(k'z - \omega t)\hat{\mathbf{x}} , \tag{4.40}$$

$$\mathbf{B}(z,t) = B_0 e^{-k''z} \cos(k'z - \omega t + \phi)\hat{\mathbf{y}} , \tag{4.41}$$

where the amplitudes of the fields are related by

$$\frac{B_0}{E_0} = \frac{|k|}{\omega} = \sqrt{\varepsilon\mu\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}} . \tag{4.42}$$

These fields are shown in Figure 4.1. We note that this description breaks at high frequencies (optical and above, $\omega \gg 10^{15} s^{-1}$, because electric conductivity itself becomes a function of ω. This case will be discussed below in section 4.5.

4.2 Reflection from a conducting surface

Now we consider what happens if the electromagnetic wave is reflected from a surface of conducting material. We assume that the *xy* plane forms the boundary between a non-conducting linear medium (1) and a conductor (2), as is shown in Figure 4.2. For simplicity, we assume that medium 1 is vacuum (so that $v_1 = c$), medium 2 has magnetic permeability $\mu_2 = \mu_0$, and we consider normal incidence.

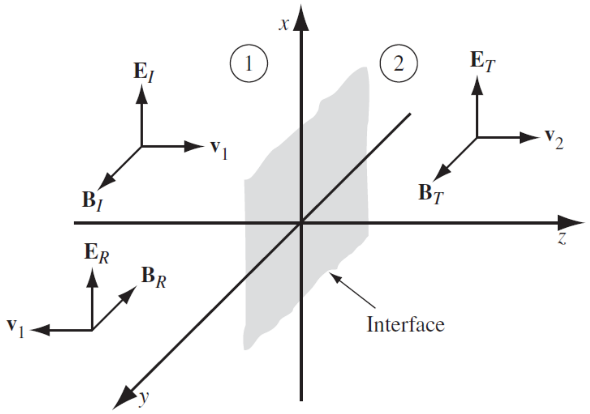

Fig. 4.2

A plane wave of frequency ω, traveling in the z direction and polarized in the x direction, approaches the interface from the left as is shown in Figure 4.2:

$$\mathbf{E}_I(z,t) = E_{0I} e^{i(k_1 z - \omega t)} \hat{\mathbf{x}}, \tag{4.43}$$

$$\mathbf{B}_I(z,t) = \frac{1}{c} E_{0I} e^{i(k_1 z - \omega t)} \hat{\mathbf{y}}. \tag{4.44}$$

It gives rise to the reflected wave

$$\mathbf{E}_R(z,t) = E_{0R} e^{i(-k_1 z - \omega t)} \hat{\mathbf{x}}, \tag{4.45}$$

$$\mathbf{B}_R(z,t) = -\frac{1}{c} E_{0R} e^{i(-k_1 z - \omega t)} \hat{\mathbf{y}}, \tag{4.46}$$

which travels back to the left in medium (1), and a transmitted wave

$$\mathbf{E}_T(z,t) = E_{0T} e^{i(k_2 z - \omega t)} \hat{\mathbf{x}}, \tag{4.47}$$

$$\mathbf{B}_T(z,t) = \frac{k_2}{\omega} E_{0T} e^{i(k_2 z - \omega t)} \hat{\mathbf{y}}, \tag{4.48}$$

which continues on the right in medium (2). Note that here, similar to the case of non-conducting medium $\mathbf{B}_0 = \frac{k}{\omega}\left(\hat{\mathbf{z}} \times \mathbf{E}_0\right)$, but due to the complex k_2, we have in Eq. (4.48) a complex number k_2/ω instead of a real $1/v_2$. Now we use boundary conditions to find the amplitudes. In the absence of surface charges and currents they have the form:

$$\varepsilon_0 E_1^{\perp} = \varepsilon_2 E_2^{\perp}, \tag{4.49}$$

$$\mathbf{E}_1^{\parallel} = \mathbf{E}_2^{\parallel}, \tag{4.50}$$

$$B_1^{\perp} = B_2^{\perp}, \tag{4.51}$$

$$\mathbf{B}_1^{\parallel} = \mathbf{B}_2^{\parallel}. \tag{4.52}$$

At $z = 0$, the combined fields on the left, $\mathbf{E}_I + \mathbf{E}_R$ and $\mathbf{B}_I + \mathbf{B}_R$, must join the fields on the right, $\mathbf{E}_T$ and $\mathbf{B}_T$, in accordance with these boundary conditions. In this case, there are no components perpendicular to the surface, so Eqs. (4.49) and (4.51) are trivial. Eqs. (4.50) and (4.52) require

$$E_{0I} + E_{0R} = E_{0T}, \tag{4.53}$$

$$\frac{1}{c}\left(E_{0I} - E_{0R}\right) = \frac{k_2}{\omega} E_{0T}. \tag{4.54}$$

Eq. (4.54) can be rewritten as follows

$$E_{0I} - E_{0R} = \beta E_{0T}, \tag{4.55}$$

where now β is equal to the *complex index of refraction n_2*:

$$\beta = n_2 = \frac{c}{\omega} k_2 . \tag{4.56}$$

Eqs. (4.53) and (4.55) are easily solved for the outgoing amplitudes, in terms of the incident amplitude:

$$E_{0R} = \frac{1-\beta}{1+\beta} E_{0I}, \quad E_{0T} = \frac{2}{1+\beta} E_{0I} . \tag{4.57}$$

These equations are formally identical to those we obtained for the reflection at the boundary between nonconductors, but the resemblance is deceptive since β now is a complex number. For a perfect conductor $\sigma = \infty$ and consequently, according to Eq. (4.39) $k_2 = \infty$, so β is infinite and

$$E_{0R} = -E_{0I}, \quad E_{0T} = 0 . \tag{4.58}$$

Therefore, the wave is totally reflected, with a 180° phase shift.

For a good conductor, $\frac{\sigma}{\varepsilon\omega} \gg 1$, the real and imaginary parts of the complex wave vector are given by Eq. (4.39). For $\mu_2 = \mu_0$ and $v_2 = c$, we obtain from Eq. (4.56) that $\beta = \gamma(1+i)$, where $\gamma \equiv \sqrt{\frac{\sigma}{2\varepsilon_0\omega}} = \frac{c}{\omega}\frac{1}{\delta}$ and δ is the skin depth (4.29). Then the reflection coefficient R takes form

$$R = \left| \frac{E_{0r}}{E_{0i}} \right|^2 = \left| \frac{1-\beta}{1+\beta} \right|^2 = \left| \frac{1-\gamma-i\gamma}{1+\gamma+i\gamma} \right|^2 = \frac{(1-\gamma)^2+\gamma^2}{(1+\gamma)^2+\gamma^2} \approx 1 - \frac{2}{\gamma} = 1 - 2\frac{\omega}{c}\delta , \tag{4.59}$$

where we assumed that $\gamma \gg 1$. This is the *Hagen-Rubens relation*, which describes the high reflectivity of metals and semiconductors at visible wavelengths. For example, for light at optical frequency $\omega = 4\times10^{15}$ s^{-1} incident from air to a silver surface ($\sigma = 6\times10^7\,\Omega^{-1}\mathrm{m}^{-1}$), we obtain $\gamma \approx 29$ and $R \approx 0.93$.

4.3 Frequency dispersion

The colored bands of a rainbow are well separated in space (dispersed) because water droplets in the atmosphere refract light with different wavelengths through different angles. Snell's law predicts this behavior because the index of refraction of water is a function of frequency. This phenomenon is called *dispersion* and the supporting medium is called *dispersive*. The frequency dispersion of the index of refraction (and other constitutive parameters) occurs because matter cannot respond instantaneously to an external perturbation.

As an example, let's consider dielectric permittivity. At static conditions, for a linear uniform dielectric, $\mathbf{D}(\mathbf{r}) = \varepsilon\mathbf{E}(\mathbf{r})$ and the response function, i.e. dielectric permittivity ε, is constant. However, when the electric field is time dependent, the response of the matter becomes *causal*. This means that the electric displacement $\mathbf{D}(\mathbf{r},t)$ depends on the electric field $\mathbf{E}(\mathbf{r},t')$ at any t' time earlier than t (but it cannot depend on the electric field at any time later than t), and the response function becomes time dependent. In this case, the linear response of the matter to the electric field, can be written as

$$\mathbf{D}(\mathbf{r},t) = \int_{-\infty}^{t} g(t-t')\mathbf{E}(\mathbf{r},t')dt' . \tag{4.60}$$

In practice, $g(\tau)\to 0$ as $\tau\to\infty$ because the electric field in the distant past has a negligible effect on the present-time electric displacement. Next, it is convenient to extend the upper limit of integration in Eq. (4.60) to infinity and build causality directly into the dielectric function. In other words, we constrain the response function in such a way that $g(\tau<0)=0$ and write

$$\mathbf{D}(\mathbf{r},t)=\int_{-\infty}^{\infty} g(t-t')\mathbf{E}(\mathbf{r},t')\,dt' . \tag{4.61}$$

The convolution integral motivates defining the Fourier transformed functions

$$g(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varepsilon(\omega)e^{-i\omega t}d\omega,\ \ \varepsilon(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(t)e^{i\omega t}dt , \tag{4.62}$$

and write Fourier representations for the electric field and displacement:

$$\mathbf{E}(\mathbf{r},t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t}d\omega,\ \ \mathbf{D}(\mathbf{r},t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{\mathbf{D}}(\mathbf{r},\omega)e^{-i\omega t}d\omega , \tag{4.63}$$

$$\tilde{\mathbf{E}}(\mathbf{r},\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathbf{E}(\mathbf{r},t)e^{i\omega t}dt,\ \ \tilde{\mathbf{D}}(\mathbf{r},\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathbf{D}(\mathbf{r},t)e^{i\omega t}dt . \tag{4.64}$$

Using these definitions and taking the Fourier transformation of Eq. (4.61) we obtain

$$\begin{aligned}\tilde{\mathbf{D}}(\mathbf{r},\omega)&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(t-t')\mathbf{E}(\mathbf{r},t')dt'e^{i\omega t}dt=\\&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathbf{E}(\mathbf{r},t')\left[\int_{-\infty}^{\infty} g(t-t')e^{i\omega(t-t')}d(t-t')\right]e^{i\omega t'}dt'=\varepsilon(\omega)\tilde{\mathbf{E}}(\mathbf{r},\omega).\end{aligned} \tag{4.65}$$

Here, the dielectric permittivity is a frequency-dependent complex linear response function. Thus, there is a direct relationship between the frequency dispersion of the dielectric response function and causality. If $\varepsilon(\omega)$ is independent of ω, Eq. (4.62) yields $g(\tau)\propto\delta(\tau)$ and, as follows from Eq. (4.60), the instantaneous connection between **D** and **E** is obtained.

Similar to the dielectric permittivity, the electric and magnetic susceptibility, magnetic permeability and electric conductivity are all frequency-dependent, for example,

$$\tilde{\mathbf{P}}(\mathbf{r},\omega)=\varepsilon_0\chi_e(\omega)\tilde{\mathbf{E}}(\mathbf{r},\omega) , \tag{4.66}$$

$$\tilde{\mathbf{J}}(\mathbf{r},\omega)=\sigma(\omega)\tilde{\mathbf{E}}(\mathbf{r},\omega) . \tag{4.67}$$

The response functions in Eqs. (4.66) and (4.67) are *not* all independent. For static fields, it is easy to distinguish the long-distance displacement of “free” charge from the short-distance displacement of “polarization” charge. This distinction becomes blurred when $\mathbf{E}(\mathbf{r},t)$ is time-harmonic because charge oscillates back and forth in both cases. At high frequency, particularly, there is simply no way to distinguish a time-harmonic conduction current with density $\mathbf{J}=\sigma\mathbf{E}$ from a time-harmonic polarization current with density $\mathbf{J}=\partial\mathbf{P}/\partial t$.

This can be seen from Ampere-Maxwell’s equation,

$$\nabla\times\mathbf{H}=\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}, \tag{4.68}$$

for a medium that obeys Ohm’s law $\mathbf{J}=\sigma\mathbf{E}$ and has dielectric constant ε_0. Assuming a *monochromatic* wave oscillating with an angular frequency ω, i.e. setting $\mathbf{E}(\mathbf{r},t)=\tilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t}$ with similar expressions for **H**, **J**, and **D**, this equation can be written in the frequency domain, as follows

$$\nabla\times\tilde{\mathbf{H}}=-i\omega\left(\varepsilon_0+i\frac{\sigma}{\omega}\right)\tilde{\mathbf{E}}. \tag{4.69}$$

If, on the other hand, we did not insert Ohm's law explicitly but attributed instead all the properties of the medium to the dielectric constant, we would identify the quantity in brackets of Eq. (4.69) with $\varepsilon(\omega)$, i.e.

$$\varepsilon(\omega)=\varepsilon_0+i\frac{\sigma(\omega)}{\omega}. \tag{4.70}$$

Therefore, it is a matter of taste whether one uses a dielectric function description or a conductivity description for time-dependent electromagnetic problems in matter.

4.4 Drude model of conducting matter

Now we discuss a simple model for frequency-dependent conductivity. Ohm's law,

$$\mathbf{J}=\sigma\mathbf{E}, \tag{4.71}$$

describes the motion of charged particles that are accelerated by an electric field but suffer energy and momentum changing collisions (known as scattering events) with other particles in the system. For example, in metals electrons suffer collisions defects and impurities. The linear dependence of **J** on **E** is one consequence of these collisions. Qualitatively, this can be seen from a classical derivation (due to Drude) which balances the electric Coulomb force on a typical conduction electron in a conductor with an effective drag force due to collisions. If τ is the average time between collisions known as the *relaxation time*, this balance makes it possible for the particles to achieve a terminal velocity **v** determined by

$$m\frac{d\mathbf{v}}{dt}=e\mathbf{E}-\frac{m\mathbf{v}}{\tau}=0. \tag{4.72}$$

The steady solution of Eq. (4.72) is called the *drift velocity*.

$$\mathbf{v}_d=\frac{e\tau}{m}\mathbf{E}. \tag{4.73}$$

The current density is determined by the drift velocity as follows:

$$\mathbf{J}=en_e\mathbf{v}_d, \tag{4.74}$$

where n_e is the density of the charged particles (electrons) and e is their charge (positive because the current direction is determined by the flow of a positive charge). Substituting (4.74) into (4.73) gives Ohm's law with

$$\sigma_0=\frac{n_e e^2\tau}{m}. \tag{4.75}$$

This expression represents the classical Drude model for the electrical conductivity in a neutral system composed of free charges. Interestingly, proper quantum mechanical calculations for a metal give the same result. The relaxation time can be determined empirically from experimental data on the conductivity. For copper, $n\simeq 8\times10^{28}$ free electrons per m^3 and at room temperature the *dc* conductivity is $\sigma_0\simeq 6\times10^7$ $\Omega^{-1}m^{-1}$. This gives $\tau\simeq 2\times10^{-14}$ s.

Now, we extend the Drude model to the case when a time-harmonic electric field $\sim\exp(-i\omega t)$ applied to a conductor producing a time-harmonic current and thus a complex frequency-dependent conductivity

$\sigma(\omega)$. To find it, we generalize our previous analysis by letting $\mathbf{E} \to \mathbf{E}(\omega)\exp(-i\omega t)$, where $\mathbf{E}(\omega)$ is a complex vector. $\mathbf{E}(\omega)$ can be considered a Fourier component of a more general time-dependent electric field. (Here and below, for simplification of the notation we denote the Fourier components of all the fields with no tilde.) Substituting it in Newton's equation of motion (4.72), we obtain

$$m\frac{d\mathbf{v}}{dt} = -e\mathbf{E}(\omega)e^{-i\omega t} - \frac{m\mathbf{v}}{\tau} . \tag{4.76}$$

Ignoring transients, a time-harmonic solution for $\mathbf{v}(t)$ is

$$\mathbf{v}(t) = \mathbf{v}(\omega)e^{-i\omega t} = \frac{e\mathbf{E}(\omega)/m}{1/\tau - i\omega}e^{-i\omega t} . \tag{4.77}$$

The amplitude of the time-harmonic current density that develops in the system is then

$$\mathbf{J}(\omega) = en_e\mathbf{v}(\omega) = \frac{n_e e^2\tau}{m}\frac{\mathbf{E}(\omega)}{1-i\omega\tau} = \sigma(\omega)\mathbf{E}(\omega) . \tag{4.78}$$

Equation (4.78) defines the complex frequency dependent Drude conductivity as

$$\sigma(\omega) = \frac{n_e e^2\tau}{m}\frac{1}{1-i\omega\tau} = \frac{\sigma_0}{1-i\omega\tau} . \tag{4.79}$$

More sophisticated calculations give the same form of $\sigma(\omega)$ for simple metals.

The dielectric function (relative frequency-dependent dielectric permittivity) is then given by

$$\varepsilon_r(\omega) = \frac{\varepsilon(\omega)}{\varepsilon_0} = \varepsilon_r'(\omega) + i\varepsilon_r''(\omega) = \left(1 - \frac{\omega_p^2\tau^2}{1+\omega^2\tau^2}\right) + i\left(\frac{\omega_p^2\tau}{\omega}\frac{1}{1+\omega^2\tau^2}\right), \tag{4.80}$$

where

$$\varepsilon_r'(\omega) = 1 - \frac{\omega_p^2\tau^2}{1+\omega^2\tau^2}, \quad \varepsilon_r''(\omega) = \frac{\omega_p^2\tau}{\omega}\frac{1}{1+\omega^2\tau^2} , \tag{4.81}$$

and we introduced the characteristic *plasma frequency*

$$\omega_p^2 \equiv \frac{n_e e^2}{\varepsilon_0 m} . \tag{4.82}$$

The physical meaning of the plasma frequency will be discussed below.

4.5 Wave propagation within the Drude model

Using the Drude model for dielectric function (4.80), let's revisit the propagation of electromagnetic waves in conducting media and reflection from a conducting surface at normal incidence. Using Eq. (4.70), we can rewrite the complex wave number given by Eq. (4.19) as follows:

$$k = \frac{\omega}{c}\sqrt{\varepsilon_r(\omega)} , \tag{4.83}$$

where we assumed for simplicity $\mu = \mu_0$. Conventionally, this equation is written as $k = \frac{\omega}{c}n$ using the *complex index of refraction n* which is defined as

$$n = n' + in'' \equiv \sqrt{\varepsilon_r(\omega)} . \tag{4.84}$$

The real part of n, n' is the *refractive index* and the imaginary part n'' is the *extinction coefficient*.

As before, the (real) electric field is given by

$$\mathbf{E}(z,t) = E_0 e^{-\frac{z}{\delta}} \cos\left[\omega\left(z - vt\right)\right]\hat{\mathbf{x}}\,, \tag{4.85}$$

where the decay length is $\delta = \dfrac{c}{\omega n''}$ and the wave velocity $v = \dfrac{c}{n'}$ is reduced as compared to that in vacuum by a factor of n'. Squaring Eq. (4.84) gives

$$n^2 = n'^2 - n''^2 + 2in'n'' = \varepsilon_r = \varepsilon_r' + i\varepsilon_r''\,, \tag{4.86}$$

so that

$$\varepsilon_r' = n'^2 - n''^2\,, \quad \varepsilon_r'' = 2n'n''\,. \tag{4.87}$$

Using these equations, one can find the refractive index and the extinction coefficient in terms of ε_r:

$$n' = \sqrt{\frac{\left|\varepsilon_r\right| + \varepsilon_r'}{2}}\,, \quad n'' = \sqrt{\frac{\left|\varepsilon_r\right| - \varepsilon_r'}{2}}\,, \tag{4.88}$$

where $\left|\varepsilon_r\right| = \sqrt{\varepsilon_r'^2 + \varepsilon_r''^2}$.

All the above considerations are valid for any dispersive material with arbitrary frequency dependent dielectric function. Now we focus on the Drude model where any $\varepsilon_r(\omega)$ is determined by Eq. (4.80).

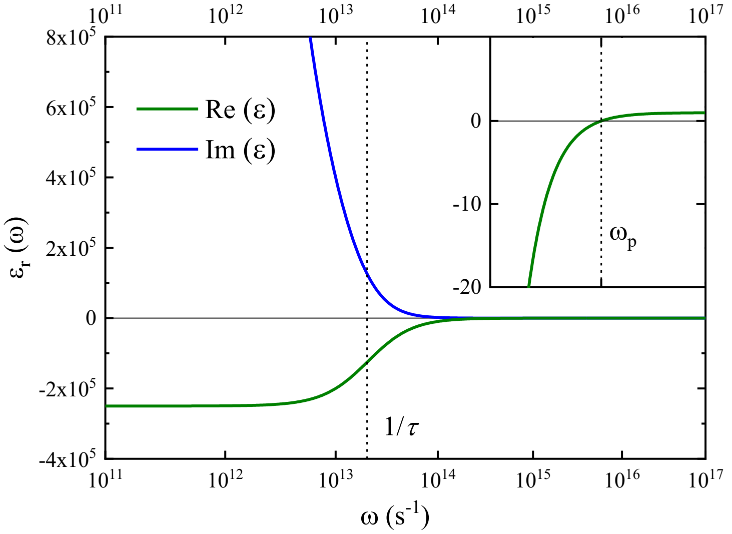

Fig. 4.3

Figure 4.3 shows the calculated real ($\varepsilon_r' = \mathrm{Re}\,\varepsilon_r$) and imaginary ($\varepsilon_r'' = \mathrm{Im}\,\varepsilon_r$) parts of the dielectric function as a function of frequency. In the calculation we assumed parameters typical for metals like copper and silver, i.e. $\omega_p = 10^{16}\,s^{-1}$ (corresponding to the ultraviolet spectrum) and $\tau = 5\times10^{-14}\,s$.

As seen from Figure 4.3, at low frequencies, the real part of the dielectric function is negative and large in magnitude. In this limit ($\omega\tau \ll 1$), as follows from Eq. (4.81), $\varepsilon_r' \approx -\omega_p^2\tau^2$ is independent of frequency. The imaginary part of the dielectric constant goes as $\varepsilon_r''(\omega) \approx \omega_p^2\tau/\omega$ and diverges at $\omega \to 0$, as expected for any conductor. For good conductors where $\omega \sim 10^{-14}$ s, this region spans the entire frequency range up to the far infrared. All the discussion of the propagation and reflection of electromagnetic waves in conductive media which was presented in Section 4.1 is valid in the low frequency region.

As the frequency gets larger and becomes compatible with $1/\tau$ ($\omega\tau \sim 1$), both ε_r' with ε_r'' are reduced in magnitude. At high frequencies ($\omega\tau \gg 1$), the imaginary part of the dielectric function drops down very quickly, as $\varepsilon_r'' \approx \frac{\omega_p^2}{\omega^3\tau}$. On the contrary, the real part of the dielectric function remains finite:

$$\varepsilon_r(\omega) \approx 1 - \frac{\omega_p^2}{\omega^2}. \tag{4.89}$$

When frequency crosses the plasma frequency ω_p, the dielectric function changes sign as seen in the inset of Figure 4.3 and tends to 1 at higher frequencies. This crossover is responsible for the anomalies in the dielectric properties and the reflection coefficient as discussed next.

The complex index of refraction $n = \sqrt{\varepsilon_r}$ directly governs wave propagation in the dispersive media. Figure 4.4 shows the real part (the refractive index) and the imaginary part (the extinction coefficient) of the complex index of refraction within the Drude metal. These are calculated using the same parameters as for the dielectric function.

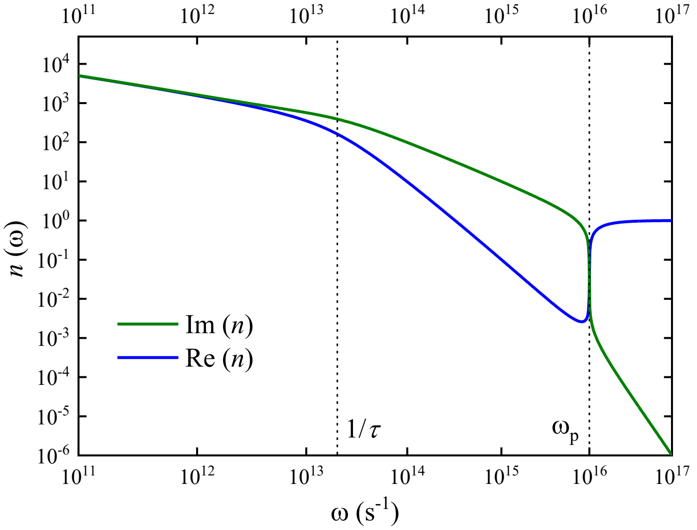

Fig. 4.4

It is seen that at low frequencies ($\omega\tau \ll 1$), n' and n'' are both large and close in magnitude. As follows from Eq. (4.88), in this limit $n' \approx n'' \approx \sqrt{\frac{\varepsilon_r''}{2}} = \sqrt{\frac{\omega_p^2\tau}{2\omega}}$. In the intermediate frequency regime ($\omega\tau \sim 1$), both n' and n'' decrease with ω but n' falls faster than n''. At higher frequencies ($\omega\tau \gg 1$), the refractive index becomes less than one ($n' < 1$). This behavior holds up to the plasma frequency ($\omega < \omega_p$). In this regime, the phase velocity, $v = \frac{c}{n'}$, exceeds the speed of light in vacuum c. This peculiar result does not violate any relativistic notion, because a monochromatic wave does not carry information (see Sec. 4.6). In this regime, $n'' > 1$ (see Fig. 4.4), and the electromagnetic wave is strongly damped; with the penetration depth, $\delta = \frac{c}{\omega n''} = \frac{\lambda_0}{2\pi}\frac{1}{n''}$, smaller than the vacuum wavelength λ_0.

At the plasma frequency, n'' falls rapidly and n' jumps up to a value close to 1. A further increase in frequency leads to further reduction of n'', whereas $n' \to 1$. The behavior of n is easy to understand based on Eq. (4.89), implying that

$$n = \begin{cases} 0 + i\sqrt{\dfrac{\omega_p^2}{\omega^2} - 1}, & \omega < \omega_p, \\ \sqrt{1 - \dfrac{\omega_p^2}{\omega^2}} + i0, & \omega > \omega_p. \end{cases} \tag{4.90}$$

Below the plasma frequency, $n = in''$ is purely imaginary, while above the plasma frequency, $n = n'$ is purely real and has high-frequency limit $n = 1$.

This behavior explains the frequency-dependent reflection coefficient R in metals. The expression for the reflection coefficient of an electromagnetic wave from a conducting surface at normal incidence can be represented as follows (see Section 4.2, where $\beta = n$):

$$R = \left|\frac{1-n}{1+n}\right|^2 = \left|\frac{1-n'-in''}{1+n'+in''}\right|^2 = \frac{(n'-1)^2 + n''^2}{(n'+1)^2 + n''^2} = 1 - \frac{4n'}{(n'+1)^2 + n''^2}. \tag{4.91}$$

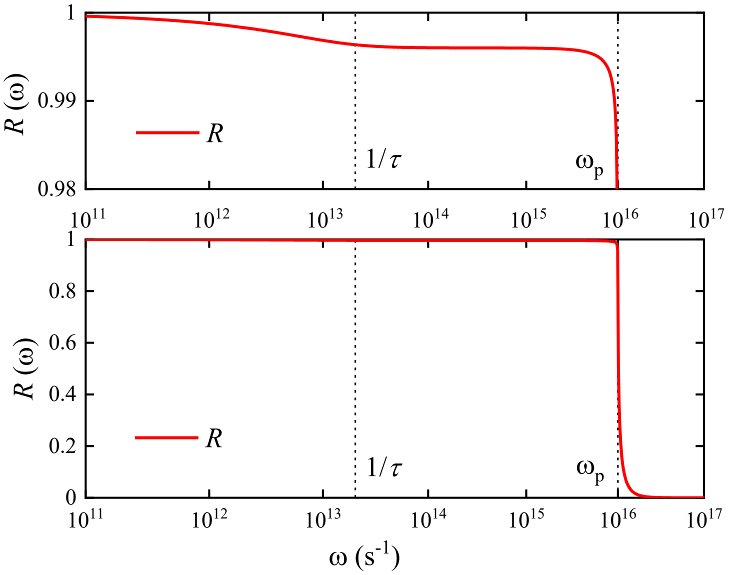

Fig. 4.5

Figure 4.5 shows $R(\omega)$ calculated for the same parameters as Figs. 4.3 and 4.4. At low frequencies ($\omega\tau \ll 1$), $n' \approx n'' \gg 1$ and we obtain as before (see Section 3.9) the Hagen-Rubens relation:

$$R \approx 1 - \frac{2}{n'} \approx 1 - 2\sqrt{\frac{2\varepsilon_0\omega}{\sigma_0}}. \tag{4.92}$$

The reflection coefficient is 100% at $\omega = 0$ and falls in accord with Eq. (4.92) with increasing frequency. Once ω becomes larger than $1/\tau$, the reflection coefficient flattens out at $R \approx 99.5\%$, as seen from the top panel of Figure 4.5.

At the plasma frequency, *R* exhibits the dramatic discontinuous drop from a value close to unity to a value close to zero. This *plasma reflection edge* occurs at the same frequency where the rapid fall of n'' and rise of n' occur in Figure 4.4. We see that above the plasma reflection edge, the metal acts like a non-absorbing transparent dielectric, e.g., glass. In metals, the plasma frequency ω_p falls into the high visible or ultraviolet range (Table 1).

Table 1: Reflection edges (plasma frequencies) and corresponding wavelengths for some metals

	Li	**Na**	**K**	**Al**	**Cu**	**Ag**
ω_p (10^{16}s^{-1})	1.22	0.89	0.59	2.40	1.64	1.36
ω_p (nm)	155	210	315	79	115	138

The nature of this charge density wave, known as a plasma oscillation, can be understood in terms of a simple model. Imagine displacing the entire electron gas, as a whole, through a distance *x* with respect to the fixed positive background of the ions (Fig. 4.6). The resulting surface charge gives rise to an electric field of magnitude $E = \sigma / \varepsilon_0$, where $\sigma = n_e e x$ is the charge per unit area at either end of the slab. This field tends to restore the electron distribution to equilibrium.

The equation of motion of the electron gas as a whole is

$$m\frac{d^2x}{dt^2} = -eE = -\frac{n_e e^2 x}{\varepsilon_0}, \tag{4.93}$$

which lead to oscillation at the plasma frequency $\omega_p^2 = \dfrac{ne^2}{\varepsilon_0 m}$.

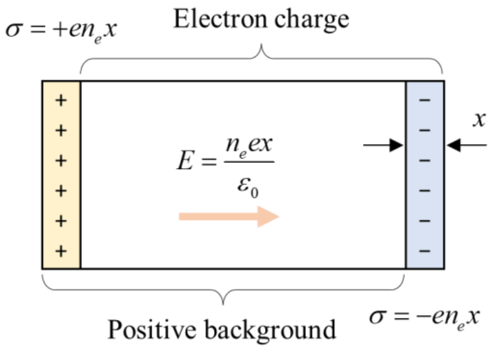

Fig. 4.6

The physical origin of the appearance of the discontinuous drop of *R* at the plasma frequency is explained by the oscillating polarization of the medium responsible for the reflected and transmitted waves. When $\omega < \omega_p$, the medium radiates two waves: a backward-propagating (reflected) wave and a forward-propagating wave which interferes destructively with the incident wave. When $\omega > \omega_p$, the oscillations of the polarization suffer a change of phase and, in place of the reflected wave, the medium creates a forward-propagating (transmitted) wave. This phase change is reminiscent of the behavior of a harmonic oscillator when is driven below and then above its resonance frequency.

4.6 Lorentz model for dielectric matter

Unlike metals, dielectrics do not have free electrons and hence require a different approach to describe their dielectric response. This section introduces the Lorenz model – a simple classical model which accounts for the frequency dependence of the dielectric function in systems with bound electrons.

We suppose that the medium is sufficiently dilute that the electric field felt by any given molecule is just the applied field **E** itself, the field exerted by other molecules being negligible. We assume that in the absence of an applied field, the molecular dipole moment is zero, at least on average. When an electric field is applied, however, some bound electrons move away from their equilibrium positions, and a dipole moment develops.

The displacement of bound charge q from its equilibrium position is denoted $\boldsymbol{\xi}$. This charge undergoes a motion driven by the applied electric field (which tends to drive $\boldsymbol{\xi}$ away from zero) and the intra-molecular forces (which tend to drive the charge back to its equilibrium position). We model these forces as harmonic forces, all sharing the same natural frequency ω_0. What we have, therefore, is a system of simple harmonic oscillators of natural frequency ω_0 that are driven at a frequency ω by an applied force. Since the motion of the charges will produce electromagnetic radiation, energy will gradually be removed from the oscillators, and the oscillations will be damped. In addition, the interaction between molecules and other charges in the system will also result in the damping of the oscillations. We incorporate this effect through a phenomenological damping parameter $\gamma > 0$. Assuming that the applied electric field is monochromatic, the equation of motion for the charge q is given by

$$m\left(\frac{d^2\boldsymbol{\xi}}{dt^2}+\gamma\frac{d\boldsymbol{\xi}}{dt}+\omega_0^2\boldsymbol{\xi}\right)=q\mathbf{E}e^{-i\omega t}\,. \tag{4.94}$$

Notice that the applied electric field $\mathbf{E}$ varies slowly over intra-molecular distances and is evaluated at the molecule's center of mass. To solve the equations of motion in the steady-state regime we write $\boldsymbol{\xi}(t)=\boldsymbol{\xi}e^{-i\omega t}$ and substitute this into Eq. (4.94). This yields

$$\boldsymbol{\xi}=\frac{q/m}{\omega_0^2-\omega^2-i\gamma\omega}\mathbf{E}\,. \tag{4.95}$$

The induced dipole moment of a molecule is $\mathbf{p}=q\boldsymbol{\xi}$. Assuming that medium has N molecules per macroscopic unit volume we find for the polarization vector

$$\mathbf{P}=\frac{q^2N}{m}\frac{1}{\omega_0^2-\omega^2-i\gamma\omega}\mathbf{E}\,. \tag{4.96}$$

Since $\mathbf{P}=(\varepsilon-\varepsilon_0)\mathbf{E}$, we obtain for the complex relative dielectric permittivity

$$\varepsilon_r(\omega)\equiv\frac{\varepsilon(\omega)}{\varepsilon_0}=1+\frac{\omega_b^2}{\omega_0^2-\omega^2-i\gamma\omega}\,. \tag{4.97}$$

where we defined the characteristic medium's frequency

$$\omega_b\equiv\left(\frac{q^2N}{\varepsilon_0 m}\right)^{1/2}. \tag{4.98}$$

This frequency is associated with bound charges and should be regarded just as some constant.

Equation (4.97) is the result of a simplistic classical model. But a proper quantum-mechanical treatment would produce a similar result, except for the fact that the electrons actually oscillate with a discrete spectrum of natural frequencies ω_j and damping coefficients ω_j. A more realistic expression for the dielectric constant is then

$$\varepsilon_r(\omega)=1+\omega_b^2\sum_j\frac{f_j}{\omega_j^2-\omega^2-i\gamma_j\omega}\,. \tag{4.99}$$

where f_j is the fraction of electrons that share the same natural frequency ω_j.

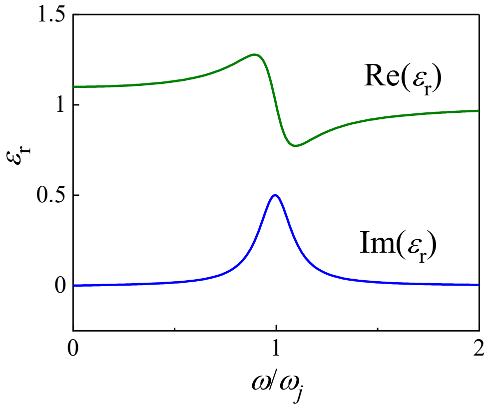

Fig. 4.7. Real and imaginary parts of the dielectric constant, as functions of ω/ω_j.

Figure 4.7 shows the real and imaginary parts of the dielectric constant around the resonant frequency ω_j. The implications of this frequency dependence are discussed in the next section. It is seen that for most frequencies, when ω is not close to ω_j, $\mathrm{Re}\,\varepsilon_r$ increases with increasing ω, and $\mathrm{Im}\,\varepsilon_r$ is very small; this typical behavior is associated with *normal dispersion*. When ω is very close to ω_j, however, the electrons are driven at almost their natural frequency, and resonance occurs. Near resonance, then, $\mathrm{Re}\,\varepsilon_r$ decreases with increasing ω, and $\mathrm{Im}\,\varepsilon_r$ is no longer small; this behavior is associated with *anomalous dispersion.* At this frequency the imaginary part of ε_r becomes large. As we will see in Sec. 4.7, a positive $\mathrm{Im}\,\varepsilon_r$ represents dissipation of energy from the electromagnetic wave into the medium, the regions where $\mathrm{Im}\,\varepsilon_r$ is large is called *resonant absorption.*

4.7 Wave propagation within the Lorentz model

As we saw earlier, the propagation of the plane electromagnetic wave in a dispersive medium is described by the attenuated wave

$$\mathbf{E}(z,t) = \mathbf{E}_0 e^{-k''z} e^{i(k'z-\omega t)} , \tag{4.100}$$

where the complex wave number is given by

$$k \equiv k' + ik'' = \frac{\omega}{c}\sqrt{\varepsilon_r} , \tag{4.101}$$

with ε_r given by Eq. (4.99). Because the wave intensity is proportional to $E^2 \sim e^{-2k''z}$, the imaginary part of k determines quantity

$$\alpha \equiv 2k'' , \tag{4.102}$$

which is called the *absorption coefficient*. The real part of k determines the wavelength, the wave (phase) velocity, and the (real) refraction index, in the usual way:

$$\lambda = \frac{2\pi}{k'} , \qquad v = \frac{\omega}{k'} , \qquad n' = \frac{c}{\omega} k' . \tag{4.103}$$

For simplicity, we assume that the second term in Eq. (4.99) is small (which is a good approximation for gases), and approximate the square root in Eq. (4.101) by the first term in the binomial expansion,

$$k = \frac{\omega}{c}\sqrt{\varepsilon_r} \approx \frac{\omega}{c}\left[1 + \frac{\omega_b^2}{2}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j \omega}\right]. \tag{4.104}$$

Within this approximation, the refraction index is given by

$$n'(\omega) \approx \frac{1+\varepsilon_r'(\omega)}{2} = 1 + \frac{\omega_b^2}{2} \sum_j \frac{f_j\left(\omega_j^2 - \omega^2\right)}{\left(\omega_j^2 - \omega^2\right)^2 + \gamma_j{}^2 \omega^2} \quad . \tag{4.105}$$

Therefore, the frequency dependence of the refraction index mimics that of the real part of the dielectric constant shown in Figure 4.7. As we noticed before, normal dispersion, where the real part of the dielectric constant, and hence the refraction index, increase with frequency occurs everywhere except the neighborhood of the resonance frequency ω_j. The normal dispersion accounts for the well-known fact that a prism refracts blue light more strongly than red light. Near resonance, $\mathrm{Re}\,\varepsilon_r$ and hence n decrease with increasing ω. This behavior describes anomalous dispersion.

It is notable that below the resonant frequency ($\omega \ll \omega_j$) $n' \simeq 1 + \frac{\omega_b^2}{2} \sum_j \frac{f_j}{\omega_j^2} > 1$, so that $v = \frac{c}{n'} < c$. On the other hand, $n' \simeq 1 - \frac{\omega_b^2}{2\omega^2} < 1$ at high frequencies, so that $v > c$. As we noticed above, the fact that the phase velocity of the wave exceeds the speed of light in vacuum, does not violate any relativistic notion, since a monochromatic wave does not carry information. To form a signal, one must modulate the wave and superpose solutions with different frequencies. As we will see in Sec. 4.8, in such situations the wave always travels with a speed that does not exceed c.

Within this approximation, the absorption coefficient (4.102) is given by

$$\alpha(\omega) \approx \frac{\omega}{c} \varepsilon_r''(\omega) = \omega_b^2 \frac{\omega^2}{c} \sum_j \frac{f_j \gamma_j}{\left(\omega_j^2 - \omega^2\right)^2 + \gamma_j{}^2 \omega^2} . \tag{4.106}$$

Therefore, its frequency dependence mimics that of the imaginary part of the dielectric constant shown in Figure 4.7. The largest absorption occurs near resonances ($\omega = \omega_j$). At high frequencies ($\omega \gg \omega_j$), ω tends to zero and therefore the medium becomes transparent. Resonant absorption occurs in the frequency region of the anomalous dispersion and that is why the anomalous dispersion is not easy to observe in practice.

Away from resonant frequencies, the damping can be ignored, and the formula (4.105) for the index of refraction simplifies

$$n \approx 1 + \frac{\omega_b^2}{2} \sum_j \frac{f_j}{\omega_j^2 - \omega^2} \quad . \tag{4.107}$$

For most substances the natural frequencies ω_j are scattered all over the spectrum in a rather chaotic fashion. But for transparent materials, the nearest significant resonances typically lie in the ultraviolet, so that $\omega < \omega_j$. In that case,

$$\frac{1}{\omega_j^2 - \omega^2} = \frac{1}{\omega_j^2}\left(1 - \frac{\omega^2}{\omega_j^2}\right)^{-1} \approx \frac{1}{\omega_j^2}\left(1 + \frac{\omega^2}{\omega_j^2}\right) , \tag{4.108}$$

and Eq. (4.107) takes the form

$$n \approx 1 + \left(\frac{\omega_b^2}{2} \sum_j \frac{f_j}{\omega_j^2}\right) + \omega^2 \left(\frac{\omega_b^2}{2} \sum_j \frac{f_j}{\omega_j^4}\right) . \tag{4.109}$$

In terms of the vacuum wavelength ($\lambda = 2\pi c/\omega$) it can be written as

$$n = 1 + A\left(1 + \frac{B}{\lambda^2}\right) . \tag{4.110}$$

This is known as *Cauchy's formula*. Constant *A* is called the coefficient of refraction, and *B* is called the coefficient of dispersion. Cauchy's equation applies reasonably well to most gases, in the optical region.

4.8 Propagation of wave packets

So far, we have considered monochromatic waves. In practice electromagnetic waves represent a superposition of monochromatic waves with different frequencies. Even in the most monochromatic light source or the most sharply tuned radio transmitter, one deals with a finite spread of frequencies or wavelengths. This spread may originate in the finite duration of a pulse, in inherent broadening in the source, or in many other ways. This leads to several new features.

1. If the medium is dispersive (i.e., the dielectric constant is a function of frequency), the phase velocity is not the same for each frequency component of the wave. Consequently, different components of the wave travel with different speeds and tend to change phase with respect to one another.

2. In a dispersive medium the velocity of energy flow may differ greatly from the phase velocity.

In this section, we consider propagation of such waves in dispersive media. For simplicity, we consider scalar waves in only one dimension. We also assume, for simplicity, that $\varepsilon(\omega)$ is *real* (i.e., we are away from resonances and ignore dissipative effects) and the wave number is positive

$$k(\omega) = \frac{\omega}{c}\sqrt{\varepsilon_r(\omega)} . \tag{4.111}$$

A wave packet is obtained by superposing plane waves with different frequencies ω. Assuming that the waves propagate along the *z* axis, we have

$$\psi(z,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} a(\omega)e^{ik(\omega)z-i\omega t}d\omega , \tag{4.112}$$

where $\psi(z,t)$ is the wave packet representing either electric or magnetic component of the electromagnetic wave, and $a(\omega)$ is a complex amplitude (Fourier component) that determines the shape of the wave packet.

Alternatively, and more conveniently for our purposes, we can instead superpose solutions with different wave numbers *k*. Defining $\omega(k)$ to be the solution of Eq. (4.111), we can write

$$\psi(z,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)e^{ikz-i\omega(k)t}dk . \tag{4.113}$$

It is not too difficult to show that the representation of Eq. (4.112) is equivalent to that of Eq. (4.113).

The complex amplitudes $A(k)$ are determined by the initial conditions we wish to impose on $\psi(z,t)$ and its time derivative. It follows from Eq. (4.113) that

$$\psi(z,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)e^{ikz}dk , \tag{4.114}$$

and inverse Fourier's transformation gives

$$A(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \psi(z,0)e^{-ikz}dz . \tag{4.115}$$

It is seen that if $\psi(z,0)$ represents a harmonic wave $e^{ik_0 z}$ for all z, then $A(k)=\sqrt{2\pi}\delta(k-k_0)$, corresponding to a monochromatic traveling wave $\psi(z,t)=e^{ik_0 z-i\omega(k_0)t}$, as required. If, however, at $t=0$, $\psi(z,0)$ represents a finite wave packet of a length of order Δz, as shown in Figure 4.8 (top), then the amplitude $A(k)$ is not a delta function. Rather, it is a peaked function with a breadth of the order of Δk, centered around a wave number k_0, as shown in Figure 4.8 (bottom).

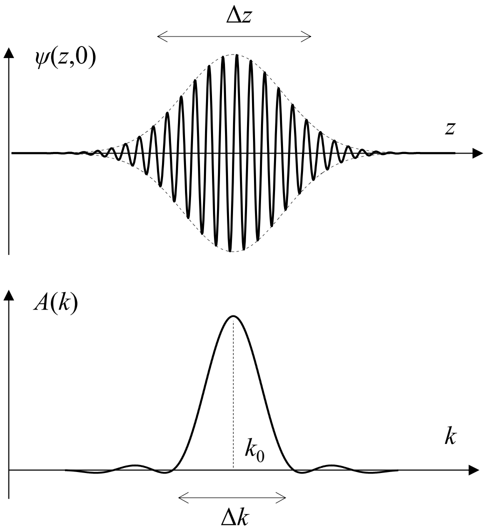

Fig. 4.8 A harmonic wave packet of a finite extent and its Fourier spectrum.

The next question is the behavior of a wave packet in time. The different wave number (frequency) components in it will move at different phase velocities. Consequently, there will be a tendency for the original coherence to be lost and for the pulse to become distorted in shape. At the very least, we might expect it to propagate with a rather different velocity from, say, the average phase velocity of its component waves.

To describe the time evolution of the pulse, we assume that the distribution $A(k)$ is fairly sharply peaked around some value $k=k_0$, as in Figure 4.8 (bottom). Suppose further that $\omega(k)$ varies slowly near $k=k_0$, so that it can be expanded around that value of k_0:

$$\omega(k)\simeq\omega(k_0)+\left.\frac{d\omega}{dk}\right|_{k=k_0}(k-k_0)=\left(\omega_0-v_g k_0\right)+v_g k\,, \tag{4.116}$$

where we denoted $\omega_0\equiv\omega(k_0)$ and defined the *group velocity*

$$v_g\equiv\left.\frac{d\omega}{dk}\right|_{k=k_0}. \tag{4.117}$$

Substituting this into Eq. (4.113), we find:

$$\psi(z,t)=\frac{1}{\sqrt{2\pi}}e^{i(k_0 v_g-\omega_0)t}\int_{-\infty}^{\infty}A(k)e^{ik(z-v_g t)}dk\,. \tag{4.118}$$

The, using Eq. (4.114), we obtain from Eq. (4.118)

$$\psi(z,t)=\psi(z-v_g t,0)e^{i(k_0 v_g-\omega_0)t}\,. \tag{4.119}$$

From Eq. (4.119), we see that in the presence of dispersion, the wave packet travels with the group velocity v_g. The phase factors indicate that the shape of the wave suffers a distortion during propagation.

The wave energy density is associated with square of its amplitude, and therefore it is clear that the transport of energy occurs with the group velocity, since that is the rate at which the pulse travels along.

For electromagnetic waves, the phase velocity is defined

$$v_p = \frac{\omega}{k} = \frac{c}{n}. \tag{4.120}$$

It is greater or smaller than c depending on whether the index of refraction n is smaller or larger than unity. To express the group velocity in terms of n, we differentiate the dispersion relation $k = \frac{n\omega}{c}$ resulting in $\frac{dk}{d\omega} = \frac{1}{c}\left(n + \omega\frac{dn}{d\omega}\right)$ and obtain

$$v_g = \frac{d\omega}{dk} = \frac{c}{n + \omega\frac{dn}{d\omega}}. \tag{4.121}$$

For normal dispersion where $dn/d\omega > 0$ and also $n > 1$, the group velocity is less than the phase velocity and also less than c. In regions of anomalous dispersion, however, $dn/d\omega$ can become large and negative as can be inferred from Figure 4.7. Then the group velocity differs greatly from the phase velocity, often becoming larger than c or even negative.

Note that the group velocity is generally not a useful concept in regions of anomalous dispersion. In addition to the existence of significant absorption, a large $dn/d\omega$ and is equivalent to a rapid variation of ω with k. Consequently, the approximations made in (4.116) and following equations are no longer valid.

As an example, we consider a conducting medium in the high-frequency region ($\omega\tau \gg 1$) above the plasma frequency so that ε is real. In this case

$$n = \sqrt{\varepsilon_r} = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}, \tag{4.122}$$

and

$$\frac{dn}{d\omega} = \frac{1}{n}\frac{\omega_p^2}{\omega^3}. \tag{4.123}$$

Using Eq. (4.121) we find

$$\frac{v_g}{c} = \frac{1}{n + \omega\frac{dn}{d\omega}} = n. \tag{4.124}$$

Since $n < 1$, we see that the group velocity $v_g = cn < c$, whereas the phase velocity $v_p = c/n > c$.

4.8 Space dispersion of a Gaussian wave packet

Here, we consider a specific model for the dependence of frequency on wave number and calculate without approximations the propagation of a wave packet in this model medium. First, we specify an initial-value problem for the wave equation. Due to being second order in time, the wave equation leads to the dispersion relation, $\omega = \omega(k)$, which is the solution of a quadratic equation, and thus allows for both positive and negative frequencies. In such a case, the solution includes two terms, which can be written as a complex amplitude and its conjugate

$$\psi(z,t)=\frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[A(k)e^{ikz-i\omega t}+A^*(k)e^{-ikz+i\omega t}\right]dk\,, \tag{4.125}$$

which can be considered as generalization of Eq. (4.113). This gives enough freedom to satisfy the two initial conditions for $\psi(z,0)$ and its time derivative $\partial\psi(z,0)/\partial t$.

$$\psi(z,0)=\frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[A(k)e^{ikz}+A^*(k)e^{-ikz}\right]dk\,, \tag{4.126}$$

$$\frac{\partial\psi}{\partial t}(z,0)=-\frac{1}{2}\frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[A(k)e^{ikz}-A^*(k)e^{-ikz}\right]\omega(k)\,dk\,. \tag{4.127}$$

Inverting these Fourier transforms gives the real and imaginary parts of $A(k)$ in terms of the initial conditions:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(z,0)e^{-ikz}dz=\frac{1}{2}\left[A(k)+A^*(-k)\right], \tag{4.128}$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial\psi}{\partial t}(z,0)e^{-ikz}dz=-\frac{i}{2}\omega(k)\left[A(k)-A^*(-k)\right]. \tag{4.129}$$

Solving these equations for $A(k)$, we find:

$$A(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\psi(z,0)+\frac{i}{\omega(k)}\frac{\partial\psi}{\partial t}(z,0)\right]e^{-ikz}dz\,. \tag{4.130}$$

Substituting this back into Eq. (4.125) gives the full-time evolution of the wave packet.

Now, let's consider the initial wave packet representing a Gaussian modulated oscillation

$$\psi(z,0)=e^{-\frac{z^2}{2\sigma^2}}\cos(k_0z)\,, \tag{4.131}$$

and, for simplicity, we assume

$$\frac{\partial\psi}{\partial t}(z,0)=0\,. \tag{4.132}$$

The latter means that at times immediately before t = 0 the wave consisted of two pulses, both moving toward the origin, such that at t = 0 they coalesced into the shape given by Eq. (4.131). Clearly at later times, we expect each pulse to reemerge on the other side of the origin. Consequently, the initial distribution is expected to split into two identical packets, one moving to the left and one the right.

The Fourier amplitude $A(k)$ is obtained from Eq. (4.130), resulting in

$$A(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{z^2}{2\sigma^2}}\frac{1}{2}\left(e^{ik_0z}+e^{-ik_0z}\right)e^{-ikz}dz=\frac{\sigma}{2}\left[e^{-\frac{\sigma^2}{2}(k-k_0)^2}+e^{-\frac{\sigma^2}{2}(k+k_0)^2}\right]. \tag{4.133}$$

To calculate the waveform at later times, we must specify ω = $\omega(k)$. As a model allowing exact calculation and showing the essential dispersive effects, we consider a conducting medium in the high-frequency region above the plasma frequency ω_p, where

$$\omega=\sqrt{\omega_p^2+c^2k^2}\approx\omega_p\left(1+\frac{c^2k^2}{2\omega_p^2}\right), \tag{4.134}$$

and we assumed for simplicity $\omega_p \gg kc$. The two pulses are expected to travel with the group velocity

$$v_g=\frac{d\omega}{dk}(k_0)=\frac{c^2k_0}{\omega_p}. \tag{4.135}$$

The exact time evolution of the wave packet is given by Eq. (4.125) with Eq. (4.133) for *A*(*k*). Due to *A*(*k*) being real, this can be written as follows:

$$\psi(z,t)=\frac{\sigma}{2\sqrt{2\pi}}\operatorname{Re}\int_{-\infty}^{\infty}\left[e^{-\frac{\sigma^2}{2}(k-k_0)^2}+e^{-\frac{\sigma^2}{2}(k+k_0)^2}\right]e^{ikz-i\omega_p\left(1+\frac{c^2k^2}{2\omega_p^2}\right)t}dk\,. \tag{4.136}$$

The two integrals differ by sign of k_0, so we consider the first one where we use new variable $k'=k-k_0$:

$$I(k_0)=\int_{-\infty}^{\infty}e^{-\frac{\sigma^2}{2}(k-k_0)^2}e^{ikz-i\omega_p\left(1+\frac{c^2k^2}{2\omega_p^2}\right)t}dk=\int_{-\infty}^{\infty}e^{-\frac{\sigma^2}{2}k'^2}e^{i(k'+k_0)z-i\omega_p t}e^{-i\omega_p\frac{c^2(k'+k_0)^2}{2\omega_p^2}t}dk'\,. \tag{4.137}$$

The integral can be performed by completing the square in the exponent:

$$\begin{aligned}I(k_0)&=\int_{-\infty}^{\infty}e^{-\left(\frac{\sigma^2}{2}+i\frac{tc^2}{2\omega_p}\right)k'^2+i\left(z-\frac{c^2k_0t}{\omega_p}\right)k'+ik_0z-i\omega_pt-i\frac{c^2k_0^2t}{2\omega_p}}dk'=\\&=\int_{-\infty}^{\infty}e^{-\left[\sqrt{\frac{\sigma^2}{2}+i\frac{tc^2}{2\omega_p}}k'-i\frac{z-\frac{c^2k_0t}{\omega_p}}{\sqrt{2\left(\sigma^2+i\frac{tc^2}{\omega_p}\right)}}\right]^2-\frac{\left(z-\frac{c^2k_0t}{\omega_p}\right)^2}{2\left(\sigma^2+i\frac{tc^2}{\omega_p}\right)}+ik_0z-i\omega_pt-i\frac{c^2k_0^2t}{2\omega_p}}dk'=\sqrt{2\pi}\left(\sigma^2+i\frac{tc^2}{\omega_p}\right)^{-1/2}e^{-\frac{\left(z-\frac{c^2k_0t}{\omega_p}\right)^2}{2\left(\sigma^2+i\frac{tc^2}{\omega_p}\right)}+ik_0z-i\omega_pt-i\frac{c^2k_0^2t}{2\omega_p}},\end{aligned} \tag{4.138}$$

where we used $\int_{-\infty}^{\infty}e^{-(\alpha k+\beta)^2}dk=\sqrt{\frac{\pi}{\alpha}}$. Using expression (4.135) for the group velocity, the final result is

$$\psi(z,t)=\frac{1}{2}\operatorname{Re}\left\{\left(1+i\frac{v_g}{\sigma^2k_0}t\right)^{-1/2}e^{-\frac{(z-v_gt)^2}{2\sigma^2\left(1+i\frac{v_g}{\sigma^2k_0}t\right)}}e^{ik_0z-i\left(\omega_p+\frac{v_gk_0}{2}\right)t}+\left(k_0\to-k_0\right)\right\}. \tag{4.139}$$

Equation (4.139) represents two pulses traveling in opposite directions. The peak amplitude of each pulse travels with the group velocity v_g, while the modulation envelope remains Gaussian in shape. The width of the Gaussian is not constant, however, but increases with time according to

$$\sigma(t)=\sqrt{\sigma^2+\left(\frac{v_g}{\sigma k_0}t\right)^2}\,. \tag{4.140}$$

It is seen that the dispersive effects on the pulse are greater (for a given elapsed time), the sharper the envelope. At long times the width of the Gaussian increases linearly with time

$$\sigma(t)\to\frac{v_g}{\sigma k_0}t\,. \tag{4.141}$$