Section 2: Conservation Laws

2.1 Conservation of energy

In this section we consider conservation of energy, often called *Poynting's theorem*.

In the EM-1 course, we found that the work necessary to assemble a static charge distribution (against the Coulomb repulsion of like charges) is given by

$$W_e = \frac{\varepsilon_0}{2}\int E^2 d^3r\,, \tag{2.1}$$

where **E** is the resulting electric field, and the integration is performed over all space.

Likewise, the work required to get currents going (against the back *emf*) is given by

$$W_m = \frac{1}{2\mu_0}\int B^2 d^3r\,, \tag{2.2}$$

where **B** is the resulting magnetic field. This suggests that the total energy stored in electromagnetic fields, per unit volume, is

$$u_{em} = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right). \tag{2.3}$$

Suppose that we have some charge and current distribution which at time t produces fields **E** and **B**. In the next instant dt the charges move around a bit. The question is how much work dw is done by the electromagnetic forces acting on these charges in the interval dt? According to the Coulomb-Lorentz force law, the work done on a charge q is

$$dw = \mathbf{F}\cdot d\mathbf{l} = q\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)\cdot\mathbf{v}dt = q\mathbf{E}\cdot\mathbf{v}dt\,, \tag{2.4}$$

which reflects the fact that the magnetic field does do work, since the magnetic force is perpendicular to the velocity. In Eq. (2.4), the charge is $q = \rho d^3r$ which results in

$$dw = \rho\mathbf{E}\cdot\mathbf{v}d^3r dt = \left(\mathbf{E}\cdot\mathbf{J}\right)d^3r dt\,, \tag{2.5}$$

where $\mathbf{J} = \rho\mathbf{v}$ is the current density. Therefore, the total rate of doing work by the fields in a finite volume V is

$$\frac{dW}{dt} = \int_V \frac{dw}{dt}d^3r = \int_V \left(\mathbf{E}\cdot\mathbf{J}\right)d^3r\,. \tag{2.6}$$

This power is delivered representing a conversion of electromagnetic energy into mechanical or thermal energy. It must be balanced by a corresponding rate of decrease of energy in the electromagnetic field within the volume V. Evidently $\mathbf{E}\cdot\mathbf{J}$ is the work done per unit time, per unit volume—which is the power delivered per unit volume. We can express this quantity in terms of the fields alone, using Ampere-Maxwell's law, $\nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}$, to eliminate **J**:

$$\mathbf{E}\cdot\mathbf{J} = \frac{1}{\mu_0}\mathbf{E}\cdot\left(\nabla\times\mathbf{B}\right) - \varepsilon_0\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}\,. \tag{2.7}$$

Using the vector identity (which comes from the cyclic permutation in the triple product),

$$\nabla\cdot\left(\mathbf{E}\times\mathbf{B}\right) = \mathbf{B}\cdot\left(\nabla\times\mathbf{E}\right) - \mathbf{E}\cdot\left(\nabla\times\mathbf{B}\right), \tag{2.8}$$

and Faraday's law, $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$, we have

$$\mathbf{E} \cdot (\nabla \times \mathbf{B}) = -\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} - \nabla \cdot (\mathbf{E} \times \mathbf{B}) . \tag{2.9}$$

Using $\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(B^2)$ and $\mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(E^2)$, we obtain

$$\mathbf{E} \cdot \mathbf{J} = -\frac{1}{2}\frac{\partial}{\partial t}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) - \frac{1}{\mu_0} \nabla \cdot (\mathbf{E} \times \mathbf{B}) . \tag{2.10}$$

Substituting this into Eq. (2.6), and applying the divergence theorem to the second term, we have

$$\frac{dW}{dt} = -\frac{d}{dt}\int_V \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) d^3r - \oint_S \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) \cdot \mathbf{n} da , \tag{2.11}$$

where S is the surface bounding V. This is *Poynting's theorem*, representing conservation of energy in electrodynamics. The first integral is the total energy stored in the electromagnetic fields. The second term represents the rate at which energy is transported out of V, across its boundary surface, by the electromagnetic fields. Poynting's theorem says, then, that the work done on the charges by the electromagnetic force is equal to the decrease in energy remaining in the fields, less the energy that flowed out through the surface.

The energy per unit time, per unit area, transported by the fields is called the *Poynting vector*:

$$\mathbf{S} \equiv \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) . \tag{2.12}$$

Specifically, $\mathbf{S} \cdot \mathbf{n} da$ is the energy per unit time crossing the infinitesimal surface da – the energy flux – so $\mathbf{S}$ is the *energy flux density*. Using the Poynting vector (2.12) and Eq. (2.3) for the electromagnetic energy density, we can express Poynting's theorem more compactly

$$\frac{dW}{dt} = -\frac{d}{dt}\int_V u_{em} d^3r - \oint_S \mathbf{S} \cdot \mathbf{n} da . \tag{2.13}$$

The work done per unit time per unit volume by the fields $(\mathbf{E} \cdot \mathbf{J})$ is a conversion of electromagnetic energy into mechanical or heat energy. Since matter is ultimately composed of charged particles (electrons and atomic nuclei), we can think of this rate of conversion as a rate of increase of *mechanical* energy (kinetic, potential, etc.) of the charged particles per unit volume. We denote this energy by u_{mech}. Then, we can interpret Poynting's theorem for the fields (**E**, **B**) as a statement of conservation of energy of the combined system of particles and fields

$$\frac{dW}{dt} = \int_V (\mathbf{E} \cdot \mathbf{J}) d^3r = \frac{d}{dt}\int_V u_{mech} d^3r . \tag{2.14}$$

Then the energy conservation law takes the form

$$\int_V \frac{\partial}{\partial t}(u_{em} + u_{mech}) d^3r = -\oint_S \mathbf{S} \cdot \mathbf{n} da = -\int_V \nabla \cdot \mathbf{S} da . \tag{2.15}$$

Since volume V is arbitrary, this can be cast into the form of a differential continuity equation

$$\frac{\partial\left(u_{em}+u_{mech}\right)}{\partial t}=-\nabla\cdot\mathbf{S}. \tag{2.16}$$

It is differential form of the Poynting's theorem. Compare it with the continuity equation expressing conservation of charge:

$$\frac{\partial\rho}{\partial t}=-\nabla\cdot\mathbf{J}. \tag{2.17}$$

The current density **J** is replaced by Pointing's vector **S**. The latter represents the flow of energy in the same way that **J** describes the flow of charge.

If no work is done on the charges in volume V (what, for example, happens in regions of empty space where there are no charges), then $\partial u_{mech}/\partial t=0$, and in this case the energy continuity equation is

$$\frac{\partial u_{em}}{\partial t}=-\nabla\cdot\mathbf{S}. \tag{2.18}$$

Example: Energy flow in a resistive wire

A long, straight, current-carrying wire with radius a and conductivity σ provides a nice illustration of Poynting's theorem (Fig. 2.1). Let the current I flow in the positive z-direction and let the wire volume be the integration volume V in Eq. (2.11). By Ohm's law, $\mathbf{J}=\sigma\mathbf{E}$, and hence the electric field is $\mathbf{E}=\frac{I}{\pi a^2\sigma}\hat{\mathbf{z}}$. According to Ampere's law, $\nabla\times\mathbf{B}=\mu_0\mathbf{J}$, resulting in the magnetic field at the wire surface $\mathbf{B}=\frac{\mu_0 I}{2\pi a}\hat{\boldsymbol{\phi}}$. According to Eq. (2.12), Poynting's vector points radially toward the center of the wire and is equal to $\mathbf{S}=-\frac{I^2}{2\pi^2a^3\sigma}\hat{\mathbf{s}}$. Moreover, because $R=\frac{L}{\pi a^2\sigma}$ is the resistance of the wire, the surface integral of Poynting's vector is given by

$$-\oint_S\mathbf{S}\cdot\mathbf{n}da=\frac{I^2L}{\pi a^2\sigma}=I^2R. \tag{2.19}$$

This confirms the conservation of energy statement (2.15). This is because for steady currents, $\partial u_{em}/\partial t=0$ due to the fields being constant in time. On the other hand, for an ohmic circuit, $\int_V\frac{\partial u_{mech}}{\partial t}d^3r=I^2R$. This follows from Eq. (2.14), as can be seen from

$$\frac{d}{dt}\int_V u_{mech}d^3r=\int_V\left(\mathbf{E}\cdot\mathbf{J}\right)d^3r=-\int_V\nabla\cdot\left(\Phi\mathbf{J}\right)d^3r+\int_V\Phi\left(\nabla\cdot\mathbf{J}\right)d^3r=\oint_S\Phi\mathbf{J}\cdot\mathbf{n}da=\left(\Phi_A-\Phi_B\right)I=I^2R, \tag{2.20}$$

where we took into account that for steady current $\nabla\cdot\mathbf{J}=0$. It also shows that a constant flow of electromagnetic energy into the wire through its side walls is required to maintain the kinetic energy of the current carrying particles against the energy they lose to ohmic heating. The energy delivered to the wire in Figure 2.1 originates from a battery or some other source of emf.

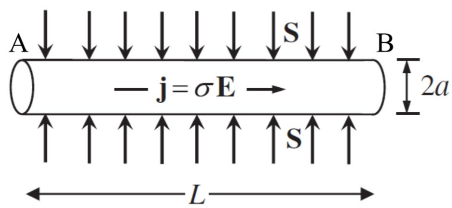

Fig. 2.1

Conservation of energy in matter

Above, we derived the energy conservation law from Maxwell's equations in vacuum. The same approach can be applied to Maxwell's equations in matter. The required algebraic steps are literally the same as for the vacuum case. In this case, the total electromagnetic energy density is given by

$$u_{em} = \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H}\right) \tag{2.21}$$

and the Poynting vector is defined by

$$\mathbf{S} = \mathbf{E}\times\mathbf{H}\,. \tag{2.22}$$

With these definitions, the Poynting theorem has the integral form (2.13) and the differential form (2.16).

For linear media, where $\mathbf{D} = \varepsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$, these equations are simplified, so that $u_{em} = \frac{1}{2}\left(\varepsilon E^2 + \frac{1}{\mu}B^2\right)$ and $\mathbf{S} = \frac{1}{\mu}\mathbf{E}\times\mathbf{B}$.

2.2 Maxwell's stress tensor

Let's calculate the total electromagnetic force on the charges in volume V:

$$\mathbf{F} = \int_V \left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)\rho d^3r = \int_V \left(\rho\mathbf{E} + \mathbf{J}\times\mathbf{B}\right)d^3r\,. \tag{2.23}$$

The force per unit volume is evidently

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{J}\times\mathbf{B}\,. \tag{2.24}$$

Now we eliminate ρ and $\mathbf{J}$ to write it in terms of the fields alone. Using Maxwell's equations, we find:

$$\mathbf{f} = \varepsilon_0\left(\nabla\cdot\mathbf{E}\right)\mathbf{E} + \left(\frac{1}{\mu_0}\nabla\times\mathbf{B} - \varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)\times\mathbf{B}\,. \tag{2.25}$$

Now

$$\frac{\partial}{\partial t}\left(\mathbf{E}\times\mathbf{B}\right) = \frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B} + \mathbf{E}\times\frac{\partial\mathbf{B}}{\partial t}\,. \tag{2.26}$$

and Faraday's law says

$$\frac{\partial\mathbf{B}}{\partial t} = -\nabla\times\mathbf{E}\,, \tag{2.27}$$

so

$$\frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B} = \frac{\partial}{\partial t}\left(\mathbf{E}\times\mathbf{B}\right) + \mathbf{E}\times\left(\nabla\times\mathbf{E}\right). \tag{2.28}$$

Thus

$$\mathbf{f} = \varepsilon_0\left(\nabla\cdot\mathbf{E}\right)\mathbf{E} - \frac{1}{\mu_0}\mathbf{B}\times\left(\nabla\times\mathbf{B}\right) - \varepsilon_0\frac{\partial}{\partial t}\left(\mathbf{E}\times\mathbf{B}\right) - \varepsilon_0\mathbf{E}\times\left(\nabla\times\mathbf{E}\right). \tag{2.29}$$

Using the product rule

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a}) + (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a}\,, \tag{2.30}$$

rule we can write

$$\nabla\left(E^2\right) = 2\left(\mathbf{E}\cdot\nabla\right)\mathbf{E} + 2\mathbf{E}\times\left(\nabla\times\mathbf{E}\right), \tag{2.31}$$

so that

$$\mathbf{E}\times\left(\nabla\times\mathbf{E}\right) = \frac{1}{2}\nabla\left(E^2\right) - \left(\mathbf{E}\cdot\nabla\right)\mathbf{E}\,. \tag{2.32}$$

Similar, we have for **B**:

$$\mathbf{B}\times(\nabla\times\mathbf{B})=\frac{1}{2}\nabla\left(B^2\right)-(\mathbf{B}\cdot\nabla)\mathbf{B}\,. \tag{2.33}$$

We can therefore rewrite Eq. (2.29) as follows

$$\begin{gathered}\mathbf{f}=\varepsilon_0(\nabla\cdot\mathbf{E})\mathbf{E}-\frac{1}{\mu_0}\left[\frac{1}{2}\nabla\left(B^2\right)-(\mathbf{B}\cdot\nabla)\mathbf{B}\right]-\varepsilon_0\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B})-\varepsilon_0\left[\frac{1}{2}\nabla\left(E^2\right)-(\mathbf{E}\cdot\nabla)\mathbf{E}\right]= \\ \varepsilon_0\left[(\nabla\cdot\mathbf{E})\mathbf{E}+(\mathbf{E}\cdot\nabla)\mathbf{E}\right]+\frac{1}{\mu_0}\left[(\mathbf{B}\cdot\nabla)\mathbf{B}+(\nabla\cdot\mathbf{B})\mathbf{B}\right]-\frac{1}{2}\left[\varepsilon_0\nabla\left(E^2\right)+\frac{1}{\mu_0}\nabla\left(B^2\right)\right]-\varepsilon_0\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B}),\end{gathered} \tag{2.34}$$

where we used $\nabla\cdot\mathbf{B}=0$. This equation can be simplified by introducing *Maxwell's stress tensor*:

$$T_{ij}\equiv\varepsilon_0\left(E_iE_j-\frac{1}{2}\delta_{ij}E^2\right)+\frac{1}{\mu_0}\left(B_iB_j-\frac{1}{2}\delta_{ij}B^2\right). \tag{2.35}$$

The indices i and j refer to the coordinates x, y, and z, so the stress tensor has a total of nine components. The Kronecker delta, δ_{ij}, is 1 if indices are the same and is zero otherwise. For example,

$$T_{xx}=\frac{1}{2}\varepsilon_0\left(E_x^2-E_y^2-E_z^2\right)+\frac{1}{2\mu_0}\left(B_x^2-B_y^2-B_z^2\right), \tag{2.36}$$

$$T_{xy}=\varepsilon_0E_xE_y+\frac{1}{\mu_0}B_xB_y\,, \tag{2.37}$$

and so on. Because of two indices sometimes tensors are denoted by a double arrow $\overleftrightarrow{\mathbf{T}}$. One can form a dot product of $\overleftrightarrow{\mathbf{T}}$ and a vector **a** in two ways—on the left, and on the right:

$$\left(\mathbf{a}\cdot\overleftrightarrow{\mathbf{T}}\right)_j=\sum_i a_iT_{ij}\,,\quad \left(\overleftrightarrow{\mathbf{T}}\cdot\mathbf{a}\right)_j=\sum_i T_{ji}a_i\,. \tag{2.38}$$

We note, however, that due to T_{kl} being symmetric, the two dot products in Eq. (2.38) are identical. The resulting object with one index is a vector. In particular, the divergence of $\overleftrightarrow{\mathbf{T}}$ has j component

$$\begin{gathered}\left(\nabla\cdot\overleftrightarrow{\mathbf{T}}\right)_j=\varepsilon_0\sum_i\nabla_i\left(E_iE_j-\frac{1}{2}\delta_{ij}E^2\right)+\frac{1}{\mu_0}\sum_i\nabla_i\left(B_iB_j-\frac{1}{2}\delta_{ij}B^2\right)= \\ \varepsilon_0\left(E_j\sum_i\nabla_iE_i+\sum_iE_i\nabla_iE_j-\frac{1}{2}\nabla_jE^2\right)+\frac{1}{\mu_0}\left(B_j\sum_i\nabla_iB_i+\sum_iB_i\nabla_iB_j-\frac{1}{2}\nabla_jB^2\right)= \\ \varepsilon_0\left[E_j(\nabla\cdot\mathbf{E})+(\mathbf{E}\cdot\nabla)E_j-\frac{1}{2}\nabla_jE^2\right]+\frac{1}{\mu_0}\left[B_j(\nabla\cdot\mathbf{B})+(\mathbf{B}\cdot\nabla)B_j-\frac{1}{2}\nabla_jB^2\right].\end{gathered} \tag{2.39}$$

Using the Maxwell stress tensor, the force per unit volume given by Eq. (2.34) can be written as:

$$\mathbf{f}=\nabla\cdot\overleftrightarrow{\mathbf{T}}-\varepsilon_0\mu_0\frac{\partial\mathbf{S}}{\partial t}, \tag{2.40}$$

where **S** is the Poynting vector. The total force on charges (Eq. (2.23)) in volume V is

$$\mathbf{F}=\int_V\left(\nabla\cdot\overleftrightarrow{\mathbf{T}}-\varepsilon_0\mu_0\frac{\partial\mathbf{S}}{\partial t}\right)d^3r=\oint_S\overleftrightarrow{\mathbf{T}}\cdot\mathbf{n}da-\varepsilon_0\mu_0\frac{d}{dt}\int_V\mathbf{S}d^3r\,, \tag{2.41}$$

where the divergence theorem was used to convert the first term to a surface integral. In the *static* case (or, more generally, whenever $\int\mathbf{S}d^3r$ is independent of time), the second term drops out, and the electromagnetic force can be expressed entirely in terms of the stress tensor at the boundary:

$$\mathbf{F} = \oint_S \overleftrightarrow{\mathbf{T}} \cdot \mathbf{n} da \ . \tag{2.42}$$

Physically, $\overleftrightarrow{\mathbf{T}}$ is the force per unit area (or *stress*) acting on the surface. More precisely, T_{ij} is the force per unit area in the i-th direction acting on an element of surface oriented in the j-th direction. The diagonal elements of the stress tensor, i.e. T_{xx}, T_{yy}, and T_{zz}, represent *compression stress* or *pressures,* and the off-diagonal elements, i.e. T_{xy}, T_{xz}, etc., are *shear stress*.

Example: Two point charges

Let us consider a simple example of the calculation of force between two equal point charges q separated by distance $2a$. The force can be calculated by constructing a plane equidistant from the two charges and integrating Maxwell's stress tensor over this plane. If we assume that the charges lie at the z axis at $+a$ and $-a$, the plane is the xy plane, and we intend to calculate the force on the upper charge, then the normal to the plane is $\mathbf{n} = -\hat{\mathbf{z}}$. Since by symmetry the force has only non-vanishing z component, we need only the zz component of the tensor:

$$T_{zz} = \varepsilon_0 \left(E_z^2 - \frac{1}{2} E^2 \right). \tag{2.43}$$

The electric field at a point on the surface placed at distance s from the z axis is

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} 2 \frac{q}{s^2 + a^2} \cos\theta \, \hat{\mathbf{s}} \, . \tag{2.44}$$

where angle θ is such that $\cos\theta = s / \sqrt{s^2 + a^2}$. It follows that $E_z = 0$ and

$$E^2 = \left(\frac{q}{2\pi\varepsilon_0} \right)^2 \frac{s^2}{(s^2 + a^2)^3} \, . \tag{2.45}$$

Therefore, we have

$$F_z = \oint_S \overleftrightarrow{\mathbf{T}} \cdot \mathbf{n} da = \frac{1}{2} \varepsilon_0 \left(\frac{q}{2\pi\varepsilon_0} \right)^2 2\pi \int_0^\infty \frac{s^2}{(s^2 + a^2)^3} s ds = \frac{q^2}{4\pi\varepsilon_0} \frac{1}{2} \int_0^\infty \frac{u du}{(u + a^2)^3} \, , \tag{2.46}$$

where we let $u \equiv s^2$. The integral can easily be taken by parts

$$\int_0^\infty \frac{u du}{(u + a^2)^3} = -\frac{1}{2} \int_0^\infty u d \left[\frac{1}{(u + a^2)^2} \right] = \frac{1}{2} \int_0^\infty \frac{du}{(u + a^2)^2} = -\frac{1}{2} \frac{1}{(u + a^2)} \bigg|_0^\infty = \frac{1}{2a^2} \, . \tag{2.47}$$

Finally, we have

$$F_z = \frac{q^2}{4\pi\varepsilon_0} \frac{1}{(2a)^2} \, . \tag{2.48}$$

Example: Charged sphere

Here, we determine the net force on the "northern" hemisphere of a uniformly charged solid sphere of radius R and charge Q.

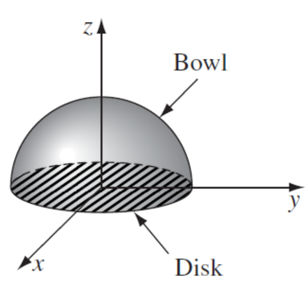

Fig. 2.2

The boundary surface consists of two parts—a hemispherical "bowl" at radius R, and a circular disk at $\theta = \pi/2$ (Fig. 2.2). For the bowl, $\mathbf{n}da = \hat{\mathbf{r}}R^2 \sin\theta d\theta d\phi$ and $\mathbf{E} = \dfrac{Q}{4\pi\varepsilon_0 R^2}\hat{\mathbf{r}}$. In Cartesian components, $\hat{\mathbf{r}} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$ and therefore

$$T_{zx} = \varepsilon_0 E_z E_x = \varepsilon_0 \left(\frac{Q}{4\pi\varepsilon_0 R^2}\right)^2 \sin\theta\cos\theta\cos\phi\,, \tag{2.49}$$

$$T_{zy} = \varepsilon_0 E_z E_y = \varepsilon_0 \left(\frac{Q}{4\pi\varepsilon_0 R^2}\right)^2 \sin\theta\cos\theta\sin\phi\,, \tag{2.50}$$

$$T_{zz} = \frac{1}{2}\varepsilon_0\left(E_z^2 - E_x^2 - E_y^2\right) = \varepsilon_0 \left(\frac{Q}{4\pi\varepsilon_0 R^2}\right)^2 \left(\cos^2\theta - \sin^2\theta\right). \tag{2.51}$$

The net force is obviously in the z-direction, so it suffices to calculate

$$\left(\overleftrightarrow{\mathbf{T}}\cdot\mathbf{n}da\right)_z = \left(T_{zx}n_x + T_{zy}n_y + T_{zz}n_z\right)da = \varepsilon_0\left(\frac{Q}{4\pi\varepsilon_0 R^2}\right)^2 R^2\cos^3\theta da\,. \tag{2.52}$$

The force on the "bowl" is therefore

$$F_{bowl} = \frac{\varepsilon_0}{2}\left(\frac{Q}{4\pi\varepsilon_0 R}\right)^2 2\pi\int_0^{\pi/2}\cos^3\theta\sin\theta d\theta = \frac{Q^2}{32\pi\varepsilon_0 R^2}\,. \tag{2.53}$$

Meanwhile, for the equatorial disk, $\mathbf{n}da = -\hat{\mathbf{z}}rdrd\phi$, and since we are inside the sphere

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 R^3}\mathbf{r} = \frac{Q}{4\pi\varepsilon_0 R^3}r\left(\cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}\right). \tag{2.54}$$

Thus

$$T_{zz} = \frac{1}{2}\varepsilon_0\left(E_z^2 - E_x^2 - E_y^2\right) = -\frac{\varepsilon_0}{2}\left(\frac{Q}{4\pi\varepsilon_0 R^3}\right)^2 r^2\,, \tag{2.55}$$

and hence

$$\left(\overleftrightarrow{\mathbf{T}}\cdot\mathbf{n}da\right)_z = \frac{\varepsilon_0}{2}\left(\frac{Q}{4\pi\varepsilon_0 R^3}\right)^2 r^3 drd\phi\,. \tag{2.56}$$

The force on the disk is therefore

$$F_{disk} = \frac{\varepsilon_0}{2}\left(\frac{Q}{4\pi\varepsilon_0 R^3}\right)^2 2\pi\int_0^R r^3 dr = \frac{Q^2}{64\pi\varepsilon_0 R^2}\,. \tag{2.57}$$

Combining Eqs. (2.53) and (2.57), we find that the net force on the northern hemisphere is

$$F = \frac{3Q^2}{64\pi\varepsilon_0 R^2}\,. \tag{2.58}$$

Incidentally, in applying Eq. (2.42), any volume that encloses all of the charge in question (and no other charge) will do the job. For example, in the present case we could use the whole region $z > 0$. In that case the boundary surface consists of the entire xy plane (plus a hemisphere at $r = \infty$, but $E = 0$ out there, so it contributes nothing). In place of the "bowl," we now have the outer portion of the plane ($r > R$). Here

$$T_{zz} = -\frac{\varepsilon_0}{2}\left(\frac{Q}{4\pi\varepsilon_0}\right)^2 \frac{1}{r^4} \tag{2.59}$$

(Eq. (2.51) with $\theta = \pi/2$ and $R \to r$), and $\mathbf{n}da = -rdr\,d\phi\,\hat{\mathbf{z}}$, so

$$\left(\overleftrightarrow{\mathbf{T}} \cdot \mathbf{n}da\right)_z = \frac{\varepsilon_0}{2}\left(\frac{Q}{4\pi\varepsilon_0}\right)^2 \frac{1}{r^3} drd\phi\,, \tag{2.60}$$

and the contribution from the plane for $r > R$ is

$$\frac{\varepsilon_0}{2}\left(\frac{Q}{4\pi\varepsilon_0}\right)^2 2\pi\int_R^\infty \frac{1}{r^3} dr = \frac{Q^2}{32\pi\varepsilon_0 R^2}\,, \tag{2.61}$$

the same as for the bowl (Eq. (2.53)).

2.3 Newton's third law in electrodynamics

Here we discuss Newton's third law in electrodynamics. Imagine a point charge q traveling along the x axis at a constant speed $\mathbf{v}$. Because it is moving, its electric field is *not* given by Coulomb's law; nevertheless, $\mathbf{E}$ still points radially outward from the instantaneous position of the charge (Fig. 2.3a). Since, moreover, a moving point charge does not constitute a steady current, its magnetic field is *not* given by the Biot-Savart law. Nevertheless, $\mathbf{B}$ still circles around the axis in a manner suggested by the right-hand rule (Fig. 2.3b).

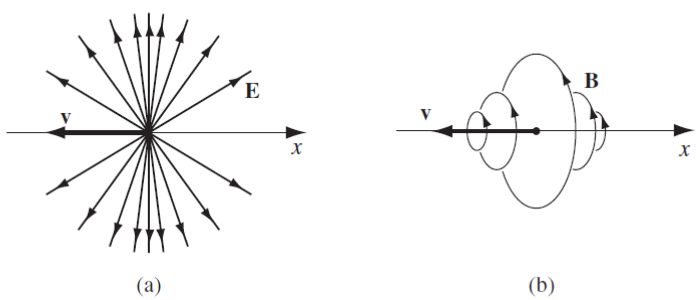

Fig. 2.3

Now suppose this moving charge encounters an identical charge, proceeding at the same speed along the y axis (Fig. 2.4). The electric force between them is repulsive, but how about the magnetic force?

The magnetic field of q_1 points into the page (at the position of q_2), so the magnetic force on q_2 is toward the *right,* whereas the magnetic field of q_2 is *out* of the page (at the position of q_1), and the magnetic force on q_1 is *upward.* This implies that the electromagnetic force of q_1 on q_2 is equal but *not* opposite to the force of q_2 on q_1 in violation of Newton's third law.

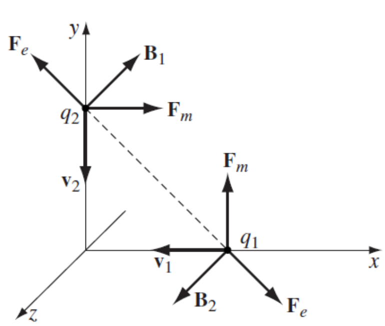

Fig. 2.4

Since forces are associated with a change of momentum in time, this fact implies that the momentum of the particles is not conserved. Momentum conservation is rescued in electrodynamics by the fact that *the fields themselves carry momentum.* In the case of the two point charges in Fig. 2.4, whatever momentum is lost to the particles is gained by the fields. Only when the field momentum is added to the mechanical momentum of the charges the momentum conservation is restored.

2.4 Conservation of momentum

According to Newton's second law, the force on an object is equal to the rate of change of its momentum:

$$\mathbf{F} = \frac{d\mathbf{P}_{mech}}{dt} . \tag{2.62}$$

Equation (2.41) can therefore be written in the form

$$\frac{d\mathbf{P}_{mech}}{dt} = -\varepsilon_0 \mu_0 \frac{d}{dt} \int_V \mathbf{S} d^3 r + \oint_S \overleftrightarrow{\mathbf{T}} \cdot \mathbf{n} da , \tag{2.63}$$

where $\mathbf{P}_{mech}$ is the total (mechanical) momentum of the particles contained in volume V. This expression is similar in structure to Poynting's theorem, and it invites an analogous interpretation. The first integral represents *momentum stored in the electromagnetic fields* themselves:

$$\mathbf{P}_{em} = \varepsilon_0 \mu_0 \int_V \mathbf{S} d^3 r , \tag{2.64}$$

while the second integral is the *momentum per unit time flowing in through the surface*. Equation (2.63) the general statement of conservation of momentum in electrodynamics: Any increase in the total momentum (mechanical plus electromagnetic) is equal to the momentum brought in by the fields:

$$\frac{d}{dt}\left(\mathbf{P}_{mech} + \mathbf{P}_{em}\right) = \oint_S \overleftrightarrow{\mathbf{T}} \cdot \mathbf{n} da . \tag{2.65}$$

If V is all of space, then no momentum flows in or out, and $\mathbf{P}_{mech}$+ $\mathbf{P}_{em}$ is constant.

Conservation of momentum can be written in a differential form. Let $\mathbf{p}_{mech}$ be the density of mechanical momentum, and $\mathbf{p}_{em}$ the density of momentum in the fields

$$\mathbf{p}_{em} = \varepsilon_0 \mu_0 \mathbf{S} . \tag{2.66}$$

Then Eq. (2.63) in the differential form says

$$\frac{\partial}{\partial t}\left(\mathbf{p}_{mech} + \mathbf{p}_{em}\right) = \nabla \cdot \overleftrightarrow{\mathbf{T}} . \tag{2.67}$$

Evidently, $-\overleftrightarrow{\mathbf{T}}$ is the *momentum flux density*, playing the role of **J** (current density) in the continuity equation, or **S** (energy flux density) in Poynting's theorem. Specifically, T_{ij} is the momentum in the i direction crossing a surface oriented in the j direction, per unit area, per unit time. Notice that Poynting's vector has appeared in two quite different roles: **S** itself is the energy per unit area, per unit time, transported by the electromagnetic fields, while $\varepsilon_0 \mu_0 \mathbf{S}$ is the momentum per unit volume stored in those fields. Similarly, $\overleftrightarrow{\mathbf{T}}$ plays a dual role: $\overleftrightarrow{\mathbf{T}}$ itself is the electromagnetic stress (force per unit area) acting on a surface, and $-\overleftrightarrow{\mathbf{T}}$ describes the flow of momentum (momentum current density) transported by the fields. Note a negative sign coming from the definition from Maxwell's stress tensor.

Example: Absorption of a transverse plane wave

A transverse plane wave is incident normally to vacuum on *perfectly absorbing* flat screen. From the law of momentum conservation, we need to show that the pressure (called radiation pressure) exerted on the screen is equal to the field energy per unit volume in the wave.

Conservation of momentum says that

$$\frac{d}{dt}\left(\mathbf{P}_{mech}+\mathbf{P}_{em}\right)=\oint_S \overleftrightarrow{\mathbf{T}}\cdot\mathbf{n}da\ , \tag{2.68}$$

The i component of the force transmitted across S and acting on the particles and field inside V.

$$\frac{d}{dt}\left(\mathbf{P}_{mech}+\mathbf{P}_{em}\right)_i=\oint_S \sum_j T_{ij}n_j da\ . \tag{2.69}$$

We take surface S to cover outside screen and going to right to contain thickness of the screen. Take the direction of propagation in $+z$ direction, with $\mathbf{n}$ as outward normal. Then, $\sum_j T_{ij}n_j$ is pressure (a force per unit area) across S into volume V. Since $\mathbf{n}=-\hat{\mathbf{z}}$, radiation pressure $\sum_j T_{ij}n_j=-T_{iz}$

The Maxwell's stress tensor is given by Eq. (2.36)

$$T_{ij}=\varepsilon_0\left(E_iE_j-\frac{1}{2}\delta_{ij}E^2\right)+\frac{1}{\mu_0}\left(B_iB_j-\frac{1}{2}\delta_{ij}B^2\right). \tag{2.70}$$

Since $E_z=B_z=0$ for the plane wave propagating along the z direction:

$$T_{iz}=\varepsilon_0\left(E_iE_z-\frac{1}{2}\delta_{iz}E^2\right)+\frac{1}{\mu_0}\left(B_iB_z-\frac{1}{2}\delta_{iz}B^2\right)=-\frac{1}{2}\delta_{iz}\left(\varepsilon_0E^2+\frac{1}{\mu_0}B^2\right). \tag{2.71}$$

Since only zz component is non-zero the radiation pressure is

$$-T_{zz}=\frac{1}{2}\left(\varepsilon_0E^2+\frac{1}{\mu_0}B^2\right)=u_{em}\ , \tag{2.72}$$

where u_{em} is the field energy density of the wave.

Example: Coaxial cable

A long coaxial cable, of length l, consists of an inner conductor (radius a) and an outer conductor (radius b). It is connected to a battery at one end and a resistor at the other so that the voltage drop between the inner and outer conductor is V (Fig. 2.5). The inner conductor carries a uniform charge per unit length λ (induced by an allied voltage V), and a steady current I to the right; the outer conductor has the opposite charge and current. What is the electromagnetic momentum stored in the fields?

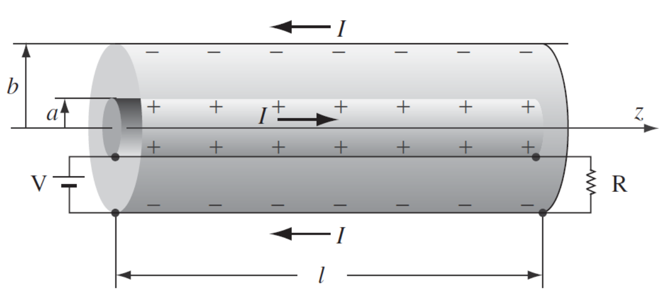

Fig. 2.5

The fields are given by

$$\mathbf{E} = \frac{\lambda}{2\pi\varepsilon_0 s}\hat{\mathbf{s}}, \quad \mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}}. \tag{2.73}$$

The Poynting vector is therefore

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} = \frac{\lambda I}{4\pi^2\varepsilon_0 s^2}\hat{\mathbf{z}}. \tag{2.74}$$

So, energy is flowing down the line, from the battery to the resistor. The power transported is

$$\Pi = \int_S \mathbf{S}\cdot\mathbf{n}da = \int_a^b \frac{\lambda I}{4\pi^2\varepsilon_0 s^2}2\pi s ds = \frac{\lambda I}{2\pi\varepsilon_0}\ln\frac{b}{a}. \tag{2.75}$$

On the other hand, the electric potential between the inner and outer conductors is

$$V = \Phi(a) - \Phi(b) = -\int_b^a \mathbf{E}\cdot d\mathbf{s} = \int_a^b \frac{\lambda}{2\pi\varepsilon_0 s}ds = \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{b}{a}. \tag{2.76}$$

Therefore, we have $\Pi = IV$ as expected.

The momentum stored in the fields is

$$\mathbf{P}_{em} = \varepsilon_0\mu_0\int_V \mathbf{S}d^3r = \varepsilon_0\mu_0\int_a^b \frac{\lambda I}{4\pi^2\varepsilon_0 s^2}\hat{\mathbf{z}}2\pi l s ds = \frac{\mu_0\lambda I}{2\pi}l\ln\frac{b}{a}\hat{\mathbf{z}} = \frac{IV}{c^2}l\hat{\mathbf{z}}. \tag{2.77}$$

While the cable is not moving, and **E** and **B** are static, there is momentum stored in the fields. Clearly, this momentum is associated with the transport of energy from the battery to the resistor.

Now, let us assume that the resistance is turned up, so the current slowly decreases. According to Faraday's law, $\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t$, the changing magnetic field induces an electric field. Integrating over the surface S bounded by a rectangular amperian loop C normal to $\hat{\boldsymbol{\phi}}$ with sides placed at s and s_0, such that $b > s > a$ and $s_0 < a$, we find

$$\int_S \nabla\times\mathbf{E}da = \oint_C \mathbf{E}\cdot d\mathbf{l} = E(s)l = -\frac{d}{dt}\int_S \mathbf{B}\cdot\mathbf{n}da = \frac{\mu_0 l}{2\pi}\frac{dI}{dt}\int_a^s \frac{1}{s'}ds' = \frac{\mu_0 l}{2\pi}\frac{dI}{dt}\ln\frac{s}{a}. \tag{2.78}$$

This field $\mathbf{E}(s) = \frac{\mu_0}{2\pi}\frac{dI}{dt}\ln\frac{s}{a}\hat{\mathbf{z}}$ exerts no force on charge $+\lambda$ at $s = a$ and the force on charge $-\lambda$ at $s = b$ is

$$\mathbf{F} = -\lambda l\mathbf{E}(b) = -\frac{\mu_0\lambda l}{2\pi}\frac{dI}{dt}\ln\frac{b}{a}\hat{\mathbf{z}}. \tag{2.79}$$

The total momentum imparted to the cable, as the current drops from I to 0, is therefore

$$\mathbf{P}_{mech} = \int \mathbf{F}dt = \frac{\mu_0\lambda I}{2\pi}l\ln\frac{b}{a}\hat{\mathbf{z}}, \tag{2.80}$$

which is precisely the momentum originally stored in the fields.

2.5 Angular momentum

In addition to energy

$$u_{em} = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right), \tag{2.81}$$

and momentum,

$$\mathbf{p}_{em} = \varepsilon_0 \mu_0 \mathbf{S} \,, \tag{2.82}$$

the electromagnetic fields are carrying the angular momentum. The angular momentum density is

$$\mathbf{l}_{em} = \mathbf{r} \times \mathbf{p}_{em} = \varepsilon_0 \mu_0 \left[\mathbf{r} \times \mathbf{S} \right]. \tag{2.83}$$

Similar to the momentum, there is a balance in the angular momentum involving mechanical and electromagnetic components. To establish the associated relationship, we proceed in a similar way to Sec. 2.2. The rate of change of angular momentum is the torque $\boldsymbol{\tau}$, which value per unit volume is determined by the rate of change of mechanical momentum density $\mathbf{l}_{mech}$ and can be written as

$$\boldsymbol{\tau} = \frac{d\mathbf{l}_{mech}}{dt} = \mathbf{r} \times \mathbf{f} = \mathbf{r} \times (\rho \mathbf{E} + \mathbf{J} \times \mathbf{B}) \,, \tag{2.84}$$

where $\mathbf{f}$ is the force per unit volume (see Eq. (2.24)). We can perform the some algebraic manipulation on the right hand side of Eq. (2.84) by using Maxwell's equations, as we did in Sec. 2.2 to obtain

$$\mathbf{f} = \nabla \cdot \overleftrightarrow{\mathbf{T}} - \frac{\partial \mathbf{p}_{em}}{\partial t}. \tag{2.85}$$

With the momentum defined by Eq. (2.82), Eq. (2.85) is identical to Eq. (2.40). We therefore obtain

$$\frac{d\mathbf{l}_{mech}}{dt} = \mathbf{r} \times \nabla \cdot \overleftrightarrow{\mathbf{T}} - \mathbf{r} \times \frac{\partial \mathbf{p}_{em}}{\partial t} = \mathbf{r} \times \nabla \cdot \overleftrightarrow{\mathbf{T}} - \frac{\partial \mathbf{l}_{em}}{\partial t} \,, \tag{2.86}$$

where $\mathbf{l}_{em}$ is the electromagnetic angular momentum density (2.83). Eq. (2.86) can be written as

$$\frac{d}{dt}(\mathbf{l}_{mech} + \mathbf{l}_{em}) = \mathbf{r} \times \nabla \cdot \overleftrightarrow{\mathbf{T}} \,. \tag{2.87}$$

This equation is reminiscent to the continuity equation for the momentum density (2.67), where $\overleftrightarrow{\mathbf{T}}$ plays a role of the momentum flux density. To have full consistency, however, we need to have the divergence of the angular momentum flux density $\overleftrightarrow{\mathcal{K}} = \mathbf{r} \times \overleftrightarrow{\mathbf{T}}$ on the right-hand side of Eq. (2.87). Here $\overleftrightarrow{\mathcal{K}}$ is the second rank tensor which is defined as follows

$$\mathcal{K}_{il} = (\mathbf{r} \times \overleftrightarrow{\mathbf{T}})_{il} = \sum_{jk} \varepsilon_{ijk} x_j T_{kl} \,, \tag{2.88}$$

where ε_{ijk} is the *Levi-Civita* symbol which is defined as follows:

$$\varepsilon_{ijk} = \begin{cases} +1, & \text{if } ikj = 123,\ 231, \text{ or } 312 \\ -1, & \text{if } ikj = 132, 213, \text{ or } 321 \\ 0, & \text{otherwise.} \end{cases} \tag{2.89}$$

Using Eq. (2.88), we can write the *i*-component of Eq. (2.87) as follows

$$\frac{d}{dt}(\mathbf{l}_{mech} + \mathbf{l}_{em})_i = \sum_{jk} \varepsilon_{ijk} x_j \sum_l \frac{\partial}{\partial x_l} T_{lk} = \sum_{jkl} \frac{\partial}{\partial x_l} (\varepsilon_{ijk} x_j T_{lk}) - \sum_{jkl} \varepsilon_{ijk} \frac{\partial x_j}{\partial x_l} T_{lk} = \left[\nabla \cdot (\mathbf{r} \times \overleftrightarrow{\mathbf{T}}) \right]_i - \sum_{lk} \varepsilon_{ilk} T_{lk} \,. \tag{2.90}$$

This last term in this equation is zero because T_{kl} is symmetric. Finally, we then obtain

$$\frac{d}{dt}(\mathbf{l}_{mech} + \mathbf{l}_{em}) = \nabla \cdot \overleftrightarrow{\mathcal{K}} \,, \tag{2.91}$$

which represents a local conservation of angular momentum. In the absence of mechanical momentum, Eq. (2.91) becomes the continuity equation for angular momentum density. We note that symmetry of the Maxwell's stress tensor is *required* by the local conservation of angular momentum.

Example: Angular momentum of a long solenoid

Imagine a very long solenoid with radius R, n turns per unit length, and current I. Coaxial with the solenoid are two long cylindrical (nonconducting) shells of length l—one, inside the solenoid at radius a, carries a charge $+Q$, uniformly distributed over its surface; the other, outside the solenoid at radius b, carries charge $-Q$ (Fig. 2.6). What happens with the shells when the current is gradually reduced?

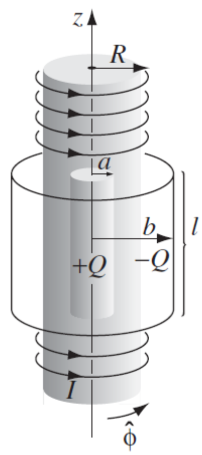

Fig. 2.6

When the current is switched off the shells start to rotate due to transfer of the angular momentum stored in the electromagnetic field to their mechanical momentum. This can be seen from the following considerations. Before the current was switched off, there was an electric field,

$$\mathbf{E} = \frac{Q}{2\pi\varepsilon_0 ls}\hat{\mathbf{s}} \quad (a < s < b), \tag{2.92}$$

in the region between the cylinders, and a magnetic field,

$$\mathbf{B} = \mu_0 nI\hat{\mathbf{z}} \quad (s < R), \tag{2.93}$$

inside the solenoid. The momentum density was therefore

$$\mathbf{p}_{em} = \varepsilon_0 \mathbf{E}\times\mathbf{B} = -\frac{\mu_0 nIQ}{2\pi ls}\hat{\boldsymbol{\phi}}\ , \tag{2.94}$$

in the region $a < s < R$. The angular momentum density was

$$\mathbf{l}_{em} = \mathbf{r}\times\mathbf{p}_{em} = -\frac{\mu_0 nIQ}{2\pi l}\hat{\mathbf{z}}\,, \tag{2.95}$$

which is constant (independent of s). To obtain the total angular momentum in the fields, we simply multiply by the volume, $\pi(R^2 - a^2)l$:

$$\mathbf{L}_{em} = -\frac{1}{2}\mu_0 nIQ(R^2 - a^2)\hat{\mathbf{z}}\,. \tag{2.96}$$

When the current is turned off, the changing magnetic field induces a circumferential electric field, given by Faraday’s law:

$$\mathbf{E} = \begin{cases} -\frac{1}{2}\mu_0 n \frac{dI}{dt}\frac{R^2}{s}\hat{\boldsymbol{\phi}}\,, & s > R, \\ -\frac{1}{2}\mu_0 n \frac{dI}{dt} s\hat{\boldsymbol{\phi}}\,, & s < R. \end{cases} \tag{2.97}$$

Thus, the torque on the outer cylinder is

$$\mathcal{T}_b = \mathbf{r} \times (-Q\mathbf{E}) = \frac{1}{2}\mu_0 n Q R^2 \frac{dI}{dt}\hat{\mathbf{z}}\,. \tag{2.98}$$

It picks up an angular momentum

$$\mathbf{L}_b = \frac{1}{2}\mu_0 n Q R^2 \hat{\mathbf{z}} \int_I^0 \frac{dI}{dt} dt = -\frac{1}{2}\mu_0 n I Q R^2 \hat{\mathbf{z}}\,. \tag{2.99}$$

Similarly, the torque on the inner cylinder is

$$\mathcal{T}_a = -\frac{1}{2}\mu_0 n Q a^2 \frac{dI}{dt}\hat{\mathbf{z}}\,, \tag{2.100}$$

and its angular momentum increase is

$$\mathbf{L}_b = \frac{1}{2}\mu_0 n I Q a^2 \hat{\mathbf{z}}\,. \tag{2.101}$$

We obtain therefore: $\mathbf{L}_{em} = \mathbf{L}_a + \mathbf{L}_b$. The angular momentum *lost* by the fields is precisely equal to the angular momentum *gained* by the cylinders, and the *total* angular momentum (fields plus matter) is conserved.