Section 1: Maxwell's Equations

1.1 Brief Summary of Electromagnetism

We start from revisiting the basic concepts of electromagnetism and summarizing the main results obtained in the first part of the Electromagnetic Theory course.

Electric Charge

An electric charge is the most fundamental quantity of electrostatics representing an intrinsic property of matter. It quantifies the strength of electrostatic interaction between charged bodies. While charge is quantized in units of the electron charge e, electromagnetic theory develops most naturally by defining a continuous charge per unit volume or *volume charge density* $\rho(\mathbf{r})$. By definition, $dq = \rho(\mathbf{r})d^3r$ is the amount of charge contained in an infinitesimal volume d^3r , so that the total charge Q in volume V is

$$Q = \int_V dq = \int_V \rho(\mathbf{r})d^3r \ . \tag{1.1}$$

In a similar way, if the continuous distribution of charge is confined to infinitesimally thin surface layers or one dimensional filaments, we can define the surface charge density $\sigma(\mathbf{r})$ or the line charge density $\lambda(\mathbf{r})$, such that the amount of charge contained in the surface area da or the line element dl are $dq = \sigma(\mathbf{r})da$ or $dq = \lambda(\mathbf{r})dl$, respectively.

A classical *point charge is defined* as a vanishingly small object which carries a finite amount of charge. The charge density of N point charges q_i located at positions $\mathbf{r}_i$ ($i = 1, 2, \ldots N$) is given by

$$\rho(\mathbf{r}) = \sum_{i=1}^{N} q_i \delta^3(\mathbf{r} - \mathbf{r}_i) \ . \tag{1.2}$$

Electric Current

Electric charge in organized motion is called electric current. A current density $\mathbf{J}(\mathbf{r}, t)$ is defined as the rate at which charge passes through an infinitesimally small area da, so that $dI = \mathbf{J}\cdot\mathbf{n}da$, where $\mathbf{n}$ is the normal to an element of surface da (Fig. 1.1 (a)). The total current that passes through a finite surface S is

$$I = \frac{dQ}{dt} = \int_S \mathbf{J}\cdot\mathbf{n}da \, . \tag{1.3}$$

We can write an explicit formula for $\mathbf{J}(\mathbf{r}, t)$ when a velocity field $\mathbf{v}(\mathbf{r}, t)$ characterizes the motion of a charge density $\rho(\boldsymbol{r}, t)$. In that case, the current density is $\mathbf{J} = \rho\mathbf{v}$.

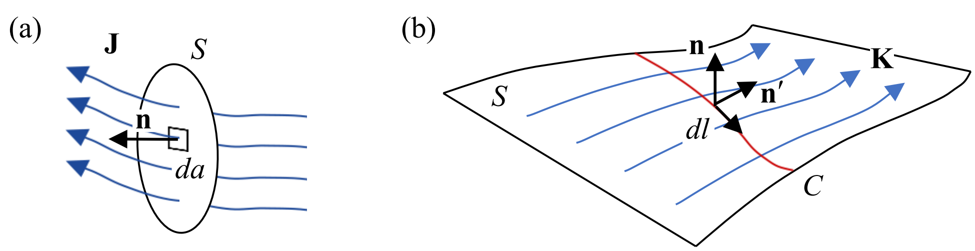

Fig. 1.1

If the charge is entirely confined to a two-dimensional surface, it is appropriate to use a surface current density $\mathbf{K} = \sigma\mathbf{v}$. Figure 1.1 (b) shows that the current passing a curve C on the surface can be expressed in terms of the local surface normal $\mathbf{n}$ using

$$I == \int_C \mathbf{dl}\cdot\mathbf{K}\times\mathbf{n} = \int_C \mathbf{K}\cdot(\mathbf{n}\times\mathbf{dl}) . \tag{1.4}$$

This makes it clear that only the projection of **K** onto the normal $\mathbf{n}'$ to the line element d**l** (in the plane of the surface) contributes to I.

Conservation of Charge

As far as we know, electric charge is absolutely conserved by all known physical processes. The only way to change the net charge in a finite volume is to move charged particles into or out of that volume.

To formulate charge conservation, we consider the surface integral of the current I in Eq. (1.3) assuming that the surface S is closed. Then the divergence theorem permits expressing I as an integral over the enclosed volume V:

$$I = \oint_S \mathbf{J}\cdot\mathbf{n}da = \int_V (\nabla\cdot\mathbf{J})d^3r . \tag{1.5}$$

Because the vector **n** in Eq. (1.3) points outward from volume V, (1.5) is the rate at which the total charge Q decreases in the volume V. An explicit expression for the latter is

$$I = -\frac{dQ}{dt} = -\frac{d}{dt}\int_V \rho d^3r = -\int_V \frac{\partial\rho}{\partial t} d^3r . \tag{1.6}$$

Equating (1.5) and (1.6) for an arbitrary volume yields a local statement of charge conservation called the *continuity equation*,

$$\nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t} . \tag{1.7}$$

The continuity equation says that the total charge in any infinitesimal volume is constant unless there is a net flow of a pre-existing charge into or out of the volume through its surface.

Example: Moving point charges

Let N point charges q_n follow trajectories $\mathbf{r}_n(t)$. The charge density of this system of moving point charges is a time-dependent generalization of Eq. (1.2)

$$\rho(\mathbf{r}) = \sum_{n=1}^{N} q_n \delta^3(\mathbf{r}-\mathbf{r}_n(t)) . \tag{1.8}$$

The particle velocities are $\mathbf{v}_n(t) = \frac{d\mathbf{r}_n}{dt} = \dot{\mathbf{r}}_n(t)$ (where the overdot denotes time derivative) and the corresponding current density is

$$\mathbf{J}(\mathbf{r},t) = \sum_{n=1}^{N} q_n \mathbf{v}_n \delta^3(\mathbf{r}-\mathbf{r}_n) . \tag{1.9}$$

Now check the continuity equation (1.7) in this case. The chain rule gives

$$\frac{\partial\rho}{\partial t} = \sum_{n=1}^{N} q_n \frac{\partial}{\partial t}\delta^3(\mathbf{r}-\mathbf{r}_n) = \sum_{n=1}^{N} q_n \sum_{i=1}^{3} \frac{\partial\delta^3(\mathbf{r}-\mathbf{r}_n)}{\partial x_{n,i}} \dot{x}_{n,i} = -\sum_{n=1}^{N} q_n \sum_{n=1}^{3} \mathbf{v}_n\cdot\nabla\delta^3(\mathbf{r}-\mathbf{r}_n) = -\nabla\cdot\sum_{n=1}^{N} q_n \mathbf{v}_n \delta^3(\mathbf{r}-\mathbf{r}_n) . \tag{1.10}$$

Here $x_{n,i}$ (i =1, 2, 3) are the Cartesian components of $\mathbf{r}_n$, i.e. $\mathbf{r}_n = (x_{n,1}, x_{n,2}, x_{n,3})$ and we took into account that $\nabla\cdot\mathbf{v}_n = 0$. According to Eq. (1.9), the right hand side of Eq. (1.10) is equal to $-\nabla\cdot\mathbf{J}$, and hence the continuity equation (1.7) holds.

Electrostatics

Coulomb's law establishes the nature of the force between stationary charged objects. Extrapolated to the case of point charges, the electrostatic force **F** on a charge q at the point **r** due to N point charges q_n located at positions $\mathbf{r}_n$ (n =1, 2, …N) is given by

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0} q \sum_{n=1}^{N} q_n \frac{\mathbf{r} - \mathbf{r}_n}{|\mathbf{r} - \mathbf{r}_n|^3} \ , \tag{1.11}$$

The pre-factor $1/4\pi\varepsilon_0$ reflects our choice of SI units. Using the point charge density (1.2), we can restate Coulomb's law in the form

$$\mathbf{F} = q\mathbf{E}(\mathbf{r}) \ , \tag{1.12}$$

where **E**(**r**) is the *electric field* given by

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0} \sum_{n=1}^{N} q_n \frac{\mathbf{r} - \mathbf{r}_n}{|\mathbf{r} - \mathbf{r}_n|^3} \ . \tag{1.13}$$

Generalizing (1.11) to a continuous charge distribution $\rho(\mathbf{r})$, we obtain for the electric field

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^3r' \ . \tag{1.14}$$

This definition makes the *principle of superposition* explicit: the electric field produced by an arbitrary charge distribution is the vector sum of the electric fields produced by each of its constituent pieces.

The associated Coulomb's force per unit volume can be written in terms of the charge density as follows

$$\mathbf{f} = \rho\mathbf{E} \ . \tag{1.15}$$

Taking divergence and curl of Eq. (1.14), we have shown earlier that

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} \rho \, , \tag{1.16}$$

$$\nabla \times \mathbf{E} = 0 \, . \tag{1.17}$$

Eq. (1.16) is *Gauss's law*, which is the direct consequence of Coulomb's law and is the first of four Maxwell's equations. Eq. (1.17) is valid for electrostatics only.

Magnetostatics

Following Oersted's discovery that a current-carrying wire produces effects qualitatively similar to those of a permanent magnet, Biot, Savart, and Ampere, performed quantitative experiments. They determined the force on a closed loop carrying a current I due to the presence of N other loops carrying currents I_n (Fig. 1.2). If **r** points to the line element $d\mathbf{l}$ of loop I and $\mathbf{r}_n$ points to the element $d\mathbf{l}_n$ of the n-th loop, Ampere's formula for the force on I is

$$\mathbf{F} = -\frac{\mu_0}{4\pi} \oint_C I d\mathbf{l} \cdot \sum_{n=1}^{N} \oint_{C_n} I_n d\mathbf{l}_n \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \ . \tag{1.18}$$

The pre-factor $\mu_0 / 4\pi$ reflects our choice of SI units.

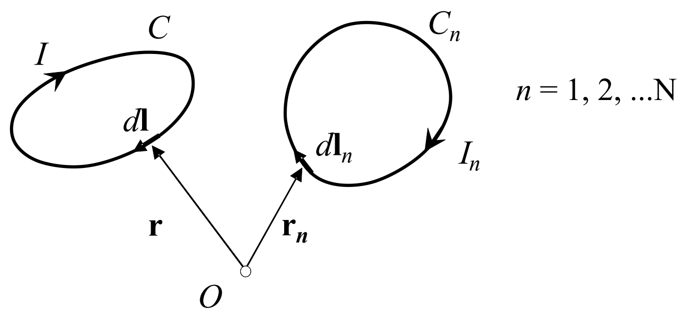

Fig. 1.2

Eq. (1.18) can be written in the form

$$\mathbf{F} = \frac{\mu_0}{4\pi}\oint_C I d\mathbf{l} \times \mathbf{B}\,, \tag{1.19}$$

where the *magnetic field* $\mathbf{B}(\mathbf{r})$ is

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\sum_{n=1}^{N}\oint_{C_n} I_n d\mathbf{l}_n \times \frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^3}\,. \tag{1.20}$$

The substitution $\int I d\mathbf{l} \Rightarrow \int \mathbf{J} d^3 r$ transforms formulae valid for line circuits into formulae valid for volume current. Accordingly, we generalize Eq. (1.20) and define the magnetic field produced by any time-independent current density $\mathbf{J}(\mathbf{r})$ as

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}') \times \left(\mathbf{r}-\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|^3} d^3 r'\,. \tag{1.21}$$

This expression is known as *Biot and Savart's law*.

The associated Ampere's force affecting solid carrying current density $\mathbf{J}(\mathbf{r})$ per unit volume is given by

$$\mathbf{f} = \mathbf{J} \times \mathbf{B}\,. \tag{1.22}$$

For moving point charges, where $\mathbf{J}$ is given by Eq. (1.9), this force is identical to the *Lorentz force*.

Taking divergence and curl of Eq.(1.21), we have shown earlier that

$$\nabla \cdot \mathbf{B} = 0\,, \tag{1.23}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}\,, \tag{1.24}$$

as long as the current density satisfies the *steady-current* condition $\nabla \cdot \mathbf{J} = 0$.

Eq. (1.23) is the second of Maxwell's equations. It has no commonly agreed-upon name but reflect the fact of absent magnetic charges. Eq. (1.24) is valid for magnetostatics only and is often called *Ampere's law*.

Faraday's Law

Faraday discovered that a transient electric current flows through a circuit whenever the magnetic flux through that circuit changes. In modern notation, Faraday's observation applied to a circuit C with resistance R implies that

$$-\frac{d}{dt}\int_S \mathbf{B} \cdot \mathbf{n}\, da = IR\,. \tag{1.25}$$

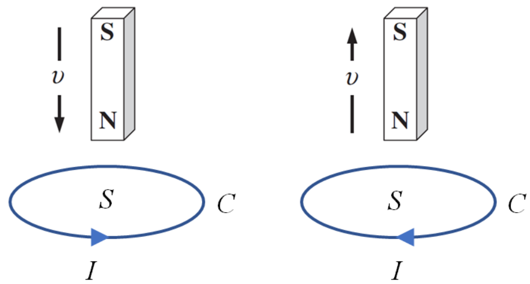

Fig. 1.3

The domain of integration S is any surface whose boundary curve coincides with the circuit C (Fig. 1.3) Our convention is that the right-hand rule relates the direction of current flow to the direction of **n**. In that case, the minus sign in Eq. (1.25) reflects Lenz' law: the current creates a magnetic field which opposes the original change in magnetic flux.

Ohm's law links the current flow I to the electromotive force $\mathcal{E}$ induced in a closed circuit C, i.e. $\mathcal{E} = IR$. On the other hand, the electromotive force is determined by an electric field **E** induced in the circuit:

$$\mathcal{E} = \oint_C \mathbf{E} \cdot d\mathbf{l}\,. \tag{1.26}$$

Therefore, after setting Eq. (1.26) equal to Eq. (1.25), Stokes' theorem yields the differential form of *Faraday's law*, the third Maxwell's equation:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}\,. \tag{1.27}$$

This relationship disregards the existence of a circuit and purely implies that a changing magnetic field induces an electric field.

Displacement Current

All the electromagnetism laws discovered *before* Maxwell can be summarized in four equations

Gauss's law: $$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}\,; \tag{1.28}$$

Absence of free magnetic poles: $$\nabla \cdot \mathbf{B} = 0\,; \tag{1.29}$$

Ampere's law: $$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}\,; \tag{1.30}$$

Faraday's law: $$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0\,. \tag{1.31}$$

It appears however that there is a fatal inconsistency in these equations. All the equations except the Faraday's law were derived from steady-state observations. However, there is no *a priopi* reason to expect that the static equations will hold unchanged for time dependent fields.

The inconsistency has to do with the rule that divergence of curl is always zero. If we apply the divergence to Eq. (1.31), everything works out:

$$\nabla \cdot (\nabla \times \mathbf{E}) = -\nabla \cdot \frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla \cdot \mathbf{B}) = 0\,. \tag{1.32}$$

The left side is zero because divergence of curl is zero; the right side is zero by virtue of Eq. (1.29). But when we do the same thing to Eq. (1.30), we get into trouble:

$$\nabla \cdot (\nabla \times \mathbf{B}) = \mu_0 \nabla \cdot \mathbf{J}\,, \tag{1.33}$$

the left side must be zero, but the right side, in general, is not. For steady currents, the divergence of **J** is zero, but evidently when we go beyond magnetostatics Ampere's law cannot be right.

Maxwell fixed this flaw by purely theoretical arguments. The problem is on the right side of Eq. (1.33), which should be zero, but isn't. Applying the continuity equation and Gauss's law, the offending term can be rewritten:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = -\varepsilon_0 \frac{\partial}{\partial t}\left(\nabla \cdot \mathbf{E}\right) = -\varepsilon_0 \nabla \cdot \frac{\partial \mathbf{E}}{\partial t}. \tag{1.34}$$

It occurs that if we were to replace **J** by $\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$, in Ampere's law, it would be just right to kill off the extra divergence. The revised Maxwell's formulation of Ampere's law is therefore:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{1.35}$$

Maxwell's Eq. (1.35) suggests that just as a changing magnetic field induces an electric field (Faraday's law), *a changing electric field induces a magnetic field*. Maxwell called his extra term the *displacement current*:

$$\mathbf{J}_D = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{1.36}$$

The confirmation of Maxwell's theory came in 1888 with Hertz's experiments on electromagnetic waves.

1.2 Maxwell's Equations in Vacuum

Classical electromagnetism summarizes a vast amount of experimental information using the concepts of charge density $\rho(\mathbf{r}, t)$, current density $\mathbf{J}(\mathbf{r}, t)$, electric field $\mathbf{E}(\mathbf{r}, t)$, and magnetic field $\mathbf{B}(\mathbf{r}, t)$ in the form of *Maxwell's equations*:

Coulomb's law (Gauss's law): $$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}\rho, \tag{1.37}$$

No name (absence of free magnetic charges): $$\nabla \cdot \mathbf{B} = 0, \tag{1.38}$$

Faraday's law: $$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1.39}$$

Ampere's law (with Maxwell's term): $$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{1.40}$$

Together with the law for the Coulomb-Lorentz force density (force per unit volume),

$$\mathbf{f} = \rho \mathbf{E} + \mathbf{J} \times \mathbf{B}, \tag{1.41}$$

they summarize the entire theoretical content of classical electrodynamics. Even the continuity equation,

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}, \tag{1.42}$$

which is the mathematical expression of the conservation of charge, can be derived from Maxwell's equations by applying the divergence to Eq. (1.40) (HW#1, Problem 1).

1.3 Duality Transformation

Maxwell's equations are asymmetric: while $\nabla \cdot \mathbf{E}$ is proportional to an electric charge density ρ, $\nabla \cdot \mathbf{B}$ is not proportional to a magnetic charge density ρ_m. Similarly, an electric current density **J** appears in the Ampere's law, but no magnetic current density $\mathbf{J}_m$ appears in Faraday's law.

This situation has led many physicists to symmetrize Maxwell's equations by introducing a *magnetic monopole* in analogy with an electric monopole (charge). Suppose that a magnetic charge exists and the motion of particles with magnetic charge produces a magnetic current density $\mathbf{J}_m$ which satisfies the continuity equation

$$\nabla \cdot \mathbf{J}_m = -\frac{\partial \rho_m}{\partial t}. \tag{1.43}$$

If we temporarily let ρ_e and $\mathbf{J}_e$ stand for the usual electric charge density and current density, these assumptions generalize Maxwell's equations to

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}\rho_e, \tag{1.44}$$

$$\nabla \cdot \mathbf{B} = \mu_0 \rho_m, \tag{1.45}$$

$$\nabla \times \mathbf{E} = -\mu_0 \mathbf{J}_m - \frac{\partial \mathbf{B}}{\partial t}, \tag{1.46}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_e + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}, \tag{1.47}$$

where we used $c^2 = 1/\mu_0\varepsilon_0$.

The new terms acquire meaning from a similarly generalized Coulomb-Lorentz force density

$$\mathbf{f} = \left(\rho_e \mathbf{E} + \mathbf{J}_e \times \mathbf{B}\right) + \left(\rho_m \mathbf{B} - \frac{1}{c^2}\mathbf{J}_m \times \mathbf{E}\right). \tag{1.48}$$

The most interesting property of the generalized Maxwell's equations is that they are invariant to a *duality transformation* of the fields and sources parameterized by an angle θ:

$$\begin{aligned} \mathbf{E}' &= \mathbf{E}\cos\theta + c\mathbf{B}\sin\theta & c\mathbf{B}' &= -\mathbf{E}\sin\theta + c\mathbf{B}\cos\theta \\ c\rho_e' &= c\rho_e\cos\theta + \rho_m\sin\theta & \rho_m' &= -c\rho_e\sin\theta + \rho_m\cos\theta \\ c\mathbf{J}_e' &= c\mathbf{J}_e\cos\theta + \mathbf{J}_m\sin\theta & \mathbf{J}_m' &= -c\mathbf{J}_e\sin\theta + \mathbf{J}_m\cos\theta. \end{aligned} \tag{1.49}$$

This means that $\mathbf{E}'$, $\mathbf{B}'$, ρ_e', ρ_m', $\mathbf{J}_e'$, and $\mathbf{J}_m'$ satisfy exactly the same equations as their unprimed counterparts. The only constraints are those imposed by the transformation itself:

$$c^2\rho_e^2 + \rho_m^2 = c^2\rho_e'^2 + \rho_m'^2. \tag{1.50}$$

Duality implies that it is strictly a matter of convention whether we say that a particle has electric charge only, magnetic charge only, or some mixture of the two. To see this, let the circle in Figure 1.4 be the locus of values of $c\rho_e$ and ρ_m permitted by (1.50). The radius vector specifies the ratio $\rho_m/c\rho_e$ for a hypothetical elementary particle with, say, electric charge $e < 0$ and magnetic charge $g > 0$. However, if the same ratio applies to every other particle in the Universe, no electromagnetic prediction changes if we exploit dual symmetry and rotate the radius vector (choose θ) to make $\rho_m = 0$ for every particle.

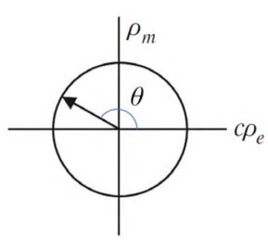

Fig. 1.4

This brings us back to the original Maxwell's equations, which are consistent with all known experiments. On the other hand, if an elementary particle is ever discovered where the intrinsic ratio *g/ce* differs from the value shown in Fig. 1.4, the option to simultaneously "rotate away" magnetic charge for all particles disappears. In that case, Eqs. (1.44) − (1.48) become the fundamental laws of Nature. This exciting possibility keeps searches for magnetic monopoles an active part of experimental physics.

1.4 Maxwell's Equations in Matter

Maxwell's equations in the form of Eqs. (1.37)-(1.40) are complete and correct as they stand. However, when we are working with materials that exhibiting electric and magnetic polarization there is a more convenient way to write them. In polarized matter, there are "bound" charges and currents, which are rapidly fluctuate on the atomic scale. Only their average over a macroscopic volume is known. It is therefore convenient to reformulate Maxwell's equations in terms of quantities **E**, **D**, **B**, **H**, **J** and ρ that are averaged over volume which include many atoms (molecules). We have already learned from the static case that an electric polarization **P** produces a bound (polarization) charge density

$$\rho_P = -\nabla \cdot \mathbf{P} . \quad (1.51)$$

Likewise, a magnetic polarization (or magnetization) **M** results in a bound (magnetization) current

$$\mathbf{J}_M = \nabla \times \mathbf{M} . \quad (1.52)$$

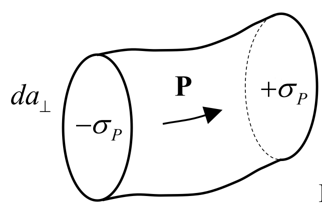

Fig. 1.5

There is one new feature to consider in the non-static case: Any *change* in the electric polarization involves a flow of bound charge (call it $\mathbf{J}_P$), which must be included in the total current. Suppose we examine a tiny chunk of polarized material (Fig. 1.5). The polarization introduces a charge density $\sigma_P = P$ at one end and $-\sigma_P$ at the other end. If P now increases a bit, the charge on each end increases accordingly, giving a net current

$$dI = \frac{\partial \sigma_P}{\partial t} da_\perp = \frac{\partial P}{\partial t} da_\perp . \quad (1.53)$$

The current density, therefore, is

$$\mathbf{J}_P = \frac{\partial \mathbf{P}}{\partial t} . \quad (1.54)$$

This polarization current $\mathbf{J}_P$ has nothing to do with the bound current $\mathbf{J}_M$ associated with magnetization of the material. The latter involves the spin and orbital motion of electrons, whereas $\mathbf{J}_P$, by contrast, is the result of the linear motion of charge when the electric polarization changes. If **P** points to the right, and is increasing, then each positive charge moves a bit to the right and each negative charge to the left, resulting in the cumulative effect of the polarization current $\mathbf{J}_P$.

The polarization current obeys the continuity equation, as seen from taking divergence of Eq. (1.54):

$$\nabla \cdot \mathbf{J}_P = \nabla \cdot \frac{\partial \mathbf{P}}{\partial t} = \frac{\partial}{\partial t}\left(\nabla \cdot \mathbf{P}\right) = -\frac{\partial \rho_P}{\partial t} . \quad (1.55)$$

Therefore, the total charge density can be separated into two parts:

$$\rho = \rho_f + \rho_P = \rho_f - \nabla \cdot \mathbf{P}, \tag{1.56}$$

and the current density into three parts:

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_M + \mathbf{J}_P = \mathbf{J}_f + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t}. \tag{1.57}$$

Gauss's law can now be written as

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}\left(\rho_f - \nabla \cdot \mathbf{P}\right), \tag{1.58}$$

or

$$\nabla \cdot \mathbf{D} = \rho, \tag{1.59}$$

where, as in the static case, the electric displacement **D** is defined by

$$\mathbf{D} \equiv \varepsilon_0 \mathbf{E} + \mathbf{P}. \tag{1.60}$$

Meanwhile, Ampère's law (with Maxwell's term) becomes

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \nabla \times \mathbf{M} + \frac{d\mathbf{P}}{dt} \right) + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \tag{1.61}$$

or

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}, \tag{1.62}$$

where, as before,

$$\mathbf{H} \equiv \frac{\mathbf{B}}{\mu_0} - \mathbf{M}, \tag{1.63}$$

and the second term in Eq. (1.62) is called the *displacement current* $\mathbf{J}_D = \frac{\partial \mathbf{D}}{\partial t}$.

Faraday's law $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ and $\nabla \cdot \mathbf{B} = 0$ are not affected by our separation of charge and current into free and bound parts, since they do not involve ρ or **J**.

In summary the macroscopic Maxwell's equations take the familiar form

Coulomb's law (Gauss law): $$\nabla \cdot \mathbf{D} = \rho, \tag{1.64}$$

Absence of free magnetic poles: $$\nabla \cdot \mathbf{B} = 0, \tag{1.65}$$

Faraday's law: $$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1.66}$$

Ampere's law (with Maxwell's term): $$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}. \tag{1.67}$$

These equations must be supplemented, by appropriate *constitutive relations*, giving **D** and **H** in terms of **E** and **B**. These depend on the nature of the material; for linear media

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E} \quad \text{and} \quad \mathbf{M} = \chi_m \mathbf{H}, \tag{1.68}$$

so that

$$\mathbf{D} = \varepsilon \mathbf{E} \quad \text{and} \quad \mathbf{B} = \mu \mathbf{H}, \tag{1.69}$$

where $\varepsilon \equiv \varepsilon_0 (1 + \chi_e)$ and $\mu \equiv \mu_0 (1 + \chi_m)$.

1.5 Boundary Conditions

In general, the fields **E**, **B**, **D**, and **H** are discontinuous at a boundary between two different media, or at a surface that carries a charge density σ_f or a current density $\mathbf{K}_f$. The explicit form of these discontinuities can be deduced from Maxwell's equations (1.64) – (1.67), in their integral form:

$$\oint_S \mathbf{D}\cdot\mathbf{n}da = \int_V \rho d^3r\,, \tag{1.70}$$

$$\oint_S \mathbf{B}\cdot\mathbf{n}da = 0\,, \tag{1.71}$$

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}\cdot\mathbf{n}da\,, \tag{1.72}$$

$$\oint_C \mathbf{H}\cdot d\mathbf{l} = \int_S \mathbf{J}\cdot\mathbf{n}da - \frac{d}{dt}\int_S \mathbf{D}\cdot\mathbf{n}da\,. \tag{1.73}$$

Applying Eq. (1.70) to a tiny Gaussian pillbox extending just slightly into the material on either side of the boundary (Fig. 1.6), we obtain:

$$\mathbf{D}_1\cdot\mathbf{n}a - \mathbf{D}_2\cdot\mathbf{n}a = \sigma_f a\,, \tag{1.74}$$

where the positive direction for **n** is assumed to be from 2 to 1. The edge of the wafer contributes nothing in the limit as the thickness goes to zero; nor does any volume charge density. Thus, the component of **D** that is perpendicular to the interface is discontinuous in the amount

$$D_1^\perp - D_2^\perp = \sigma_f\,. \tag{1.75}$$

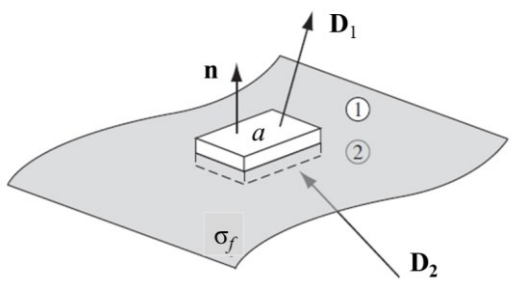

Fig. 1.6

Identical reasoning, applied to Eq. (1.71), yields

$$B_1^\perp - B_2^\perp = 0\,. \tag{1.76}$$

Turning to Eq. (1.72), a very thin amperian loop straddling the surface (Fig. 1.7) gives

$$\mathbf{E}_1\cdot\mathbf{l} - \mathbf{E}_2\cdot\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}\cdot\mathbf{n}da\,. \tag{1.77}$$

But in the limit as the width of the loop goes to zero, the flux vanishes. Therefore,

$$\mathbf{E}_1^\parallel - \mathbf{E}_2^\parallel = 0\,. \tag{1.78}$$

That is, the components of **E** parallel to the interface are continuous across the boundary.

By the same token, Eq. (1.73) implies

$$\mathbf{H}_1\cdot\mathbf{l} - \mathbf{H}_2\cdot\mathbf{l} = I_f\,, \tag{1.79}$$

where I_f is the free current passing through the amperian loop. In the limit of infinitesimal width, no *volume* current density contributes, but a *surface* current does. Noting that if **n** is a unit vector perpendicular to the interface (pointing from 2 toward 1), so that (**n** × **l**) is normal to the amperian loop (Fig. 1.7), we obtain

$$I_f = \mathbf{K}_f \cdot (\mathbf{n} \times \mathbf{l}) = (\mathbf{K}_f \times \mathbf{n}) \cdot \mathbf{l}, \tag{1.80}$$

and hence

$$\mathbf{H}_1^{\parallel} - \mathbf{H}_2^{\parallel} = \mathbf{K}_f \times \mathbf{n}. \tag{1.81}$$

We see that parallel components of **H** are discontinuous by an amount proportional to the free surface current density.

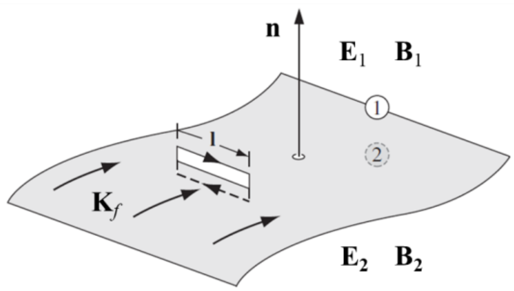

Fig. 1.7

Equations (1.75), (1.76), (1.78), and (1.79) are the general boundary conditions for electrodynamics. In the case of linear media, they can be expressed in terms of **E** and **B** alone:

$$\varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = \sigma_f, \tag{1.82}$$

$$\mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0, \tag{1.83}$$

$$B_1^{\perp} - B_2^{\perp} = 0, \tag{1.84}$$

$$\frac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \frac{1}{\mu_2}\mathbf{B}_2^{\parallel} = \mathbf{K}_f \times \mathbf{n}. \tag{1.85}$$